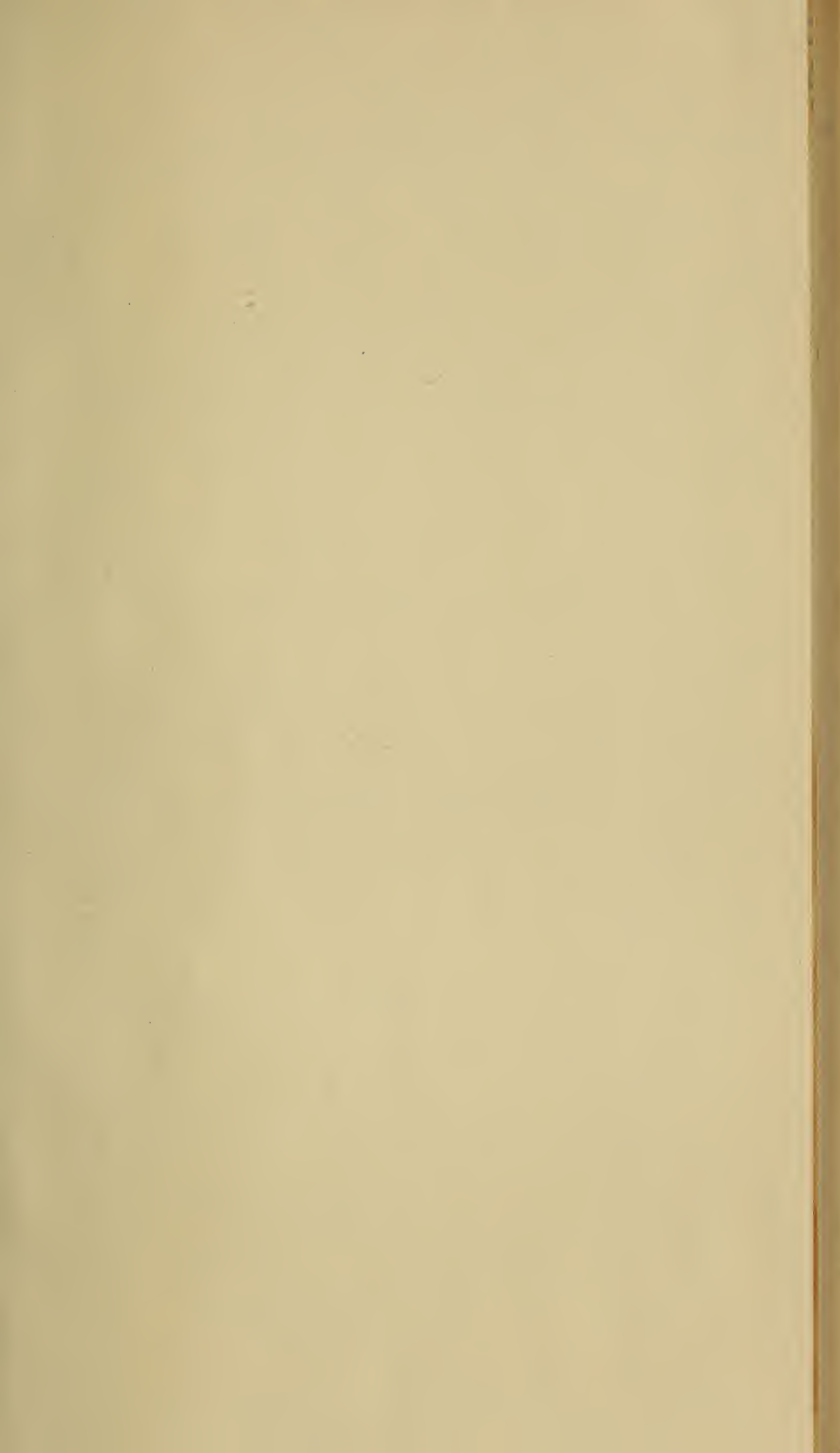


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ELEMENTS
OF
GEOMETRY;
CONTAINING
THE FIRST SIX BOOKS OF EUCLID,
WITH
TWO BOOKS
ON THE GEOMETRY OF SOLIDS.
TO WHICH ARE ADDED,
ELEMENTS OF
PLANE AND SPHERICAL
TRIGONOMETRY.

BY JOHN PLAYFAIR, F. R. S. EDIN.
Professior of Mathematics in the University of Edinburgh.

EDINBURGH:
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P R E F A C E.

IT is a remarkable fact in the history of science, that the oldest book of Elementary Geometry is still considered as the best, and that the writings of EUCLID, at the distance of two thousand years, continue to form the most approved introduction to the mathematical sciences. This peculiar distinction the Greek geometer owes to the elegance and correctness of his demonstrations, added to an arrangement most happily contrived for the purposes of instruction; advantages which, when they reach a certain eminence, secure the works of an author from being forgotten, more effectually than even originality of invention. In passing, however, through the hands of the ancient editors, during the decline of science, the excellence of his writings had been considerably obscured, and much skill and learning have been employed by the modern mathematicians to deliver them from blemishes, with which, it is certain, that they were not originally marked. Of these

mathematicians, Dr SIMSON, as he may be accounted the last, has also been the most successful, and has left very little room for the ingenuity of future editors to be exercised in, either by amending the text of EUCLID, or by improving the translations from it.

Such being the merits of Dr SIMSON's edition, and the reception it has met with having been every way suited to them, the work now offered to the public will perhaps appear unnecessary. And indeed, if that geometer, had written with a view of accommodating the Elements of EUCLID to the present state of the mathematical sciences, it is not likely that any thing new in Elementary Geometry would have been soon attempted. But his design was different; it was his object to restore the writings of EUCLID to their original perfection, and to give them to modern Europe as nearly as possible in the state wherein they made their first appearance in ancient Greece. For this undertaking no body could be better qualified than Dr SIMSON; who, to an accurate knowledge of the learned languages, and a most indefatigable spirit of research, added a profound skill in the ancient Geometry, and an admiration of it almost enthusiastic. Accordingly, he not only restored the text of EUCLID wherever it had been corrupted, but in some cases removed imperfections that probably belonged

to.

to the original work ; though his extreme partiality for his author never permitted him to suppose, that this was an honour that could fall to the share either of himself, or of any other of the moderns.

But, after all this was accomplished, something still remained to be done, since, notwithstanding the acknowledged excellence of EUCLID's Elements, it could not be doubted, that some alterations might be made upon them, that would accommodate them better to a state of the mathematical sciences, so much more improved and extended than at any former period. This accordingly is the object of the edition now offered to the public, which is intended not so much to give to the writings of Euclid the form which they originally had, as that which may at present render them most useful.

One of the alterations that has been made with this view, respects the Doctrine of Proportion, the method of treating which, as it is laid down in the fifth of EUCLID, has great advantages, accompanied with considerable defects ; of which, however, it must be observed, that the advantages are essential to it, and the defects only accidental. To explain the nature of the former, requires a more minute examination than is suited to this place, and which must, therefore, be reserved for

the notes; and, in the mean time, it may be sufficient to remark, that no definition of proportionals, except that of EUCLID, has ever been given, from which their properties can be deduced by reasonings, which, at the same time that they are perfectly rigorous, are also simple and direct. As to the defects, on the other hand, the prolixness and obscurity, that have so often been complained of in this book, they seem to arise entirely from the nature of the language; for, in mathematics, common language can seldom be applied, without much tediousness and circumlocution, in reasoning about the relations of such things as cannot be represented by means of diagrams, which happens here, where the subject treated of is magnitude in general. It is plain, therefore, that the concise language of Algebra is directly calculated to remedy this inconvenience; and such a one I have, accordingly, endeavoured to introduce, in the simplest form, and without changing at all the nature of the reasoning, or departing in any thing from the rigour of geometrical demonstration. By this contrivance the steps of the reasoning which were before so far separated, are brought near to one another, and the force of the whole is so clearly and directly perceived, that I am persuaded no more difficulty will be found in understanding the propositions

positions of the fifth Book, than of any other of the Elements.

A few changes have also been made in the enunciations of this book, chiefly in those of the subsidiary propositions which EUCLID introduced for the sake of the rest; they are expressed here in the manner that seemed best adapted to the new notation.

The alterations above mentioned are the most material that have been attempted on the books of EUCLID: There are, however, a few others, which, though less considerable, it is hoped, may in some degree facilitate the understanding of them. Such are those made on the definitions in the first Book, and particularly on that of a straight line. A new Axiom is also introduced in the room of the 12th, for the purpose of demonstrating more easily some of the properties of parallel lines. In the third Book, the remarks concerning the angles made by a straight line, and the circumference of a circle, are left out, as tending to perplex one who has advanced no farther than the elements of the science. The 27th, 28th and 29th of the 6th are changed for easier and more simple propositions, which do not materially differ from them, and which answer exactly the same purpose. Some propositions also have been added; but, for a fuller detail concerning these changes,

I must refer to the notes, in which several of the more difficult, or more interesting subjects of Elementary Geometry are treated at considerable length.

Thus much for the part of the Elements that treats of Plane Figures. With respect to the Geometry of Solids, I have departed from EUCLID altogether, with a view of rendering it both shorter and more comprehensive. This, however, is not attempted by introducing a mode of reasoning looser or less rigorous than that of the Greek geometer; for this would be to pay too dear even for the time that might thereby be saved; but it is done chiefly by laying aside a certain rule, which, though it be not essential to the accuracy of demonstration, EUCLID has thought it proper, as much as possible, to observe.

The rule referred to, is one which regulates the arrangement of EUCLID's propositions through the whole of the Elements, *viz.* That in the demonstration of a theorem he never supposes any thing to be done, as any line to be drawn, or any figure to be constructed, the manner of doing which he has not previously explained. Now, the only use of this rule is to prevent the admission of impossible or contradictory suppositions, which no doubt might lead into error; and it is a rule well calculated to answer that end; as
it

it does not allow the existence of any thing to be supposed, unless the thing itself be actually exhibited. But it is not always necessary to make use of this defence, for the existence of many things is obviously possible, and far enough from implying a contradiction, where the method of actually exhibiting them may be altogether unknown. Thus, it is plain, that on any given figure as a base, a solid may be constituted, or conceived to exist, equal to a given solid, (because a solid, whatever be its base, as its height may be indefinitely varied, is capable of all degrees of magnitude, from nothing upwards), and yet, it may in many cases be a problem of extreme difficulty to assign the height of such a solid, and actually to exhibit it. Now, this very supposition is one of those, by the introduction of which, the Geometry of Solids is much shortened, while all the real accuracy of the demonstrations is preserved; and therefore, to follow, as EUCLID has done, the rule that excludes this, and such like hypotheses, is to create artificial difficulties, and to embarrass geometrical investigation with more obstacles than the nature of things has thrown in its way. It is a rule, too, which cannot always be followed, and from which even EUCLID himself has been forced to depart, in more than one instance.

In the two Books, therefore, on the Properties of Solids, that I now offer to the public, though

I have followed EUCLID very closely in the simpler parts, I have no where sought to subject the demonstrations to such a law as the foregoing, and have never hesitated to admit the existence of such solids, or such lines as are evidently possible, though the manner of actually describing them may not have been explained. In this way also, I have been enabled to offer that very refined artifice in geometrical reasoning, to which we give the name of the Method of Exhaustions, under a much simpler form than it appears in the 12th of EUCLID; and the spirit of it may, I think, be best learned when it is disengaged from every thing not essential to it. That this method may be the better understood, and because the demonstrations that require it are, no doubt, the most difficult in the Elements, they are all conducted as nearly as possible in the same way through the different solids, from the pyramid to the sphere. The comparison of this last solid with the cylinder concludes the eight Book, and is a proposition that may not improperly be considered as terminating the elementary part of Geometry.

In the beginning of the Book just mentioned, I have treated pretty fully of the rectification and quadrature of the Circle, subjects that are often omitted altogether in works of this kind. They are omitted, however, as I conceive, without any good reason, because, to measure the length of the simplest of all the curves which Geometry treats

treats of, and the space contained within it, are problems that certainly belong to the elements of the science, especially as they are not more difficult than other propositions where the method of exhaustions is employed. When I speak of the rectification of the circle, or of measuring the length of the circumference, I must not be supposed to mean, that a straight line is to be made equal to the circumference *exactly*, a problem which, as is well known, Geometry has never been able to resolve: All that is proposed is, to determine two straight lines that differ very little from one another, not more, for instance, than the four hundred and ninety-seventh part of the diameter of the circle, and of which the one is demonstrated to be greater than the circumference of that circle, and the other to be less. In the same manner, the quadrature of the circle is performed only by approximation, or by finding two rectangles, nearly equal to one another, the one of them greater, and the other less than the space contained within the circle.

The Data of EUCLID has been annexed to several editions of the Elements, and particularly to Dr SIMSON's, but in this it is omitted altogether. It is omitted, however, not from any opinion of its being in itself useless, but because it does not belong to this place, and is not often read by beginners. It contains the rudiments of what is properly
I called

called the Geometrical Analysis, and has itself an analytical form ; and, for these reasons, I would willingly reserve it, or rather a compend of it, for a work on that analysis, which I have long meditated.

Plane and Spherical Trigonometry, on the other hand, make a part of this volume, because, in every course of mathematical studies, that is directed toward useful purposes, these two branches necessarily come after the Elements. In explaining the elements of such sciences, there is not much new that can be attempted, or that will be expected by the intelligent reader. Except, perhaps, some new demonstrations, and some changes in the arrangement, these two treatises have, accordingly, no novelty to boast of. The Plane Trigonometry, though pretty full, is so divided, that the part of it that is barely sufficient for the resolution of Triangles, may be easily taught by itself. In a scholium the method of constructing the trigonometrical Tables is explained, and a demonstration is added of the properties of the sines and co-sines of the sums and differences of arches, which are the foundation of those new applications of Trigonometry that have been introduced with so much advantage into the higher Geometry.

In the Spherical Trigonometry, the rules for preventing the ambiguity of the solutions, wherever it can be prevented, have been particularly attended to ; and I have availed myself as much as possible of that excellent abstract of the rules of this science, which Dr MASKELYNE has prefixed to the new tables of Logarithms.

It has been objected to many of the writers on Elementary Geometry, and particularly to EUCLID, that they have been at great pains to prove the truth of many simple propositions, which every body is ready to admit, without any demonstration, and thus take up the time, and fatigue the attention of the student, to no purpose. To this objection also, if there be any force in it, the present treatise is certainly as much exposed as any other, for, of all the alterations that may be made in the Elements, the last I should think of, is to consider any thing as self-evident that admits of demonstration. Indeed, those who make the objection just stated, do not seem to have reflected sufficiently on the end of Mathematical Demonstration, which is not only to prove the truth of a certain proposition, but to shew its necessary connection with other propositions, and its dependence on them. The truths of Geometry are all necessarily connected with one another, and the system of such truths can never
be

be rightly explained, unless that connection be accurately traced, wherever it exists. It is upon this that the beauty and peculiar excellence of the mathematical sciences depend; it is this that prevents any one truth from being single and insulated, and connects the different parts so firmly, that they must all stand, or all fall together. The demonstration, therefore, even of an obvious proposition, answers the purpose of connecting that proposition with others, and ascertaining its place in the general system of mathematical truth. If, for example, it be alleged, that it is needless to demonstrate that any two sides of a triangle are greater than the third; it may be replied, that this is no doubt a truth, which, without proof, most men will be inclined to admit; but, are we for that reason to account it of no consequence to know what the propositions are, which would cease to be true if this proposition were supposed to be false? Is it not useful to know, that unless it be true, that any two sides of a triangle are greater than the third, neither could it be true, that the greater side of every triangle is opposite to the greater angle, nor that the equal sides are opposite to equal angles, nor, lastly, that things equal to the same thing are equal to one another? By a scientific mind this information will not be thought lightly of; and it is exactly that which we receive from EUCLID'S demonstration.

To all this it may be added, that the mind, especially when beginning to study the art of reasoning, cannot be employed to greater advantage than in analysing those judgments, which, though they appear simple, are in reality complex, and capable of being distinguished into parts. No progress in ascending higher can be expected till a regular habit of demonstration is thus acquired; and I should greatly suspect, that he who has declined the trouble of tracing the connection between the proposition already quoted, and those that are below it, would never be very expert in tracing its connection with those that are above it; and that, as he had not been careful in laying the foundation, he would never be successful in raising the superstructure.

COLLEGE OF EDINBURGH,

Oct. 21. 1795.

ERRATA.

Book III. Definitions, for Def. XI. *read* Def. IX.

Book V. Prop. XXIV. line third of the Demonstration, *read* Because

$E : B :: F : D$, by inversion, $B : E :: C : D$.

P. 263. fourth line, for 2, *read* two.

E L E M E N T S

OF

G E O M E T R Y.

B O O K I.

D E F I N I T I O N S.

I.

A Point is that which has position, but not magnitude. See Notes.

II.

A line is length without breadth.

COROLLARY. The extremities of a line are points; and "the intersections of one line with another are also points."

III.

Lines which cannot coincide in two points, without coinciding altogether, are called straight lines.

COR. Hence two straight lines cannot inclose a space. Neither can two straight lines have a common segment; "that is, they cannot coincide in part, without coinciding altogether."

IV.

A superficies is that which hath only length and breadth.

COR. The extremities of a superficies are lines; and the "intersections of one superficies with another are also lines."

B

V.

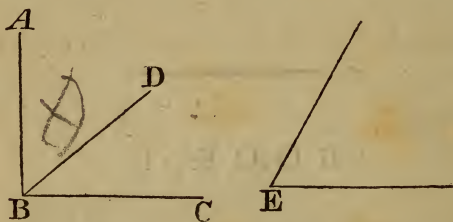
Book I.

V.

A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VI.

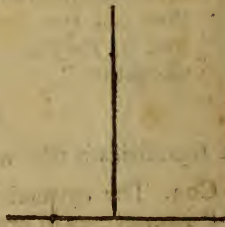
A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N. B. ‘ When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line: Thus the angle which is contained by the straight lines AB, CB, is named the angle ABC, or CBA; that which is contained by AB, BD is named the angle ABD, or DBA; and that which is contained by BD, CB is called the angle DBC, or CBD; but, if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at E.’

VII.

When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

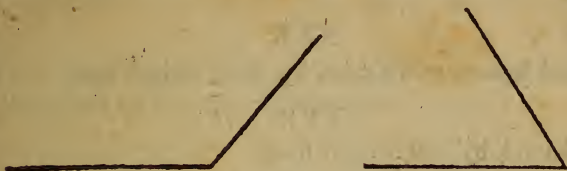


VIII.

VIII.

An obtuse angle is that which is greater than a right angle.

Book I.



IX.

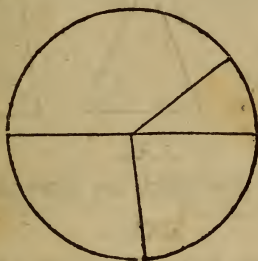
An acute angle is that which is less than a right angle.

X.

A figure is that which is inclosed by one or more boundaries.

XI.

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



XII.

And this point is called the centre of the circle.

XIII.

A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XIV.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XV.

Rectilineal figures are those which are contained by straight lines.

XVI.

Trilateral figures, or triangles, by three straight lines.

XVII.

Quadrilateral, by four straight lines.

XVIII.

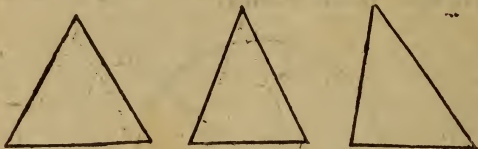
Multilateral figures, or polygons, by more than four straight lines.

XIX.

Of three sided figures, an equilateral triangle is that which has three equal sides.

XX.

An isosceles triangle is that which has only two sides equal.



XXI.

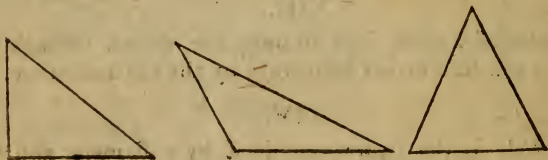
A scalene triangle, is that which has three unequal sides.

XXII.

A right angled triangle, is that which has a right angle.

XXIII.

An obtuse angled triangle, is that which has an obuse angle.



XXIV.

OF GEOMETRY.

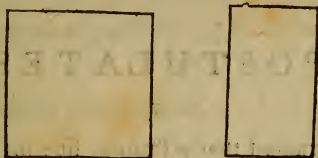
Book I.

XXIV.

An acute angled triangle, is that which has three acute angles.

XXV.

Of four sided figures, a square is that which has all its sides equal, and all its angles right angles.

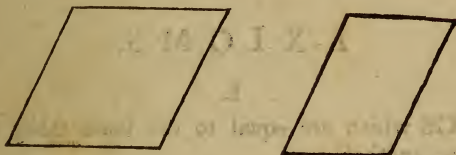


XXVI.

An oblong, is that which has all its angles right angles, but has not all its sides equal.

XXVII.

A rhombus, is that which has all its sides equal, but its angles are not right angles.



XXVIII.

A rhomboid, is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

XXIX.

All other four sided figures besides these, are called Trapeziums.

XXX.

Book I.

XXX.

Parallel straight lines, are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

POSTULATES.

I.

LET it be granted that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any centre, at any distance from that centre.

A X I O M S.

I.

THINGS which are equal to the same thing are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

All right angles are equal to one another.

XI.

“ Two straight lines cannot be drawn through the same point,
“ parallel to the same straight line, without coinciding with
“ one another.

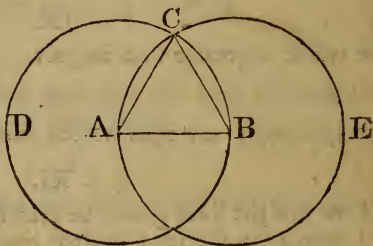
PROPO.

PROPOSITION I. PROBLEM.

TO describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line ; it is required to describe an equilateral triangle upon it.

From the centre A , at the distance AB , describe ^a the circle BCD , and from the centre B , at the distance BA , describe the circle ACE ; and from the point C , in which the circles cut one another, draw the straight lines ^b CA , CB to the points A , B : ABC shall be an equilateral triangle.



Because the point A is the centre of the circle BCD , AC is equal ^c to AB ; and because the point B is the centre of the circle ACE , BC is equal to BA : But it has been proved that CA is equal to AB ; therefore CA , CB are each of them equal to AB ; but things which are equal to the same are equal to one another ^d; therefore CA is equal to CB ; wherefore CA , AB , BC are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB . Which was required to be done.

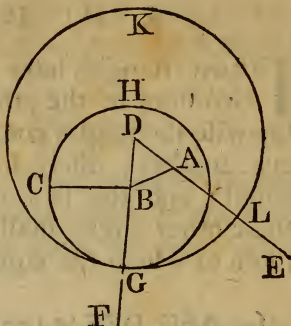
PROP. II. PROB.

FROM a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line ; it is required to draw from the point A a straight line equal to BC .

From

From the point A to B draw ^a the straight line AB; and upon it describe ^b the equilateral triangle DAB, and produce ^c the straight lines DA, DB, to E and F; from the centre B, at the distance BC, describe ^d the circle CGH, and from the centre D, at the distance DG, describe the circle GKL. AL shall be equal to BC.



Book I.
a 1. Post.
b 1. 1.
c 2. Post.

d 3. Post.

e 11. Def.

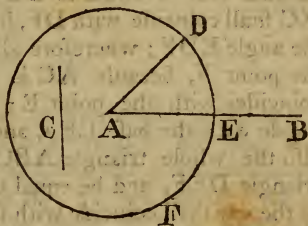
Because the point B is the centre of the circle CGH, BC is equal ^e to BG; and because D is the centre of the circle GKL, DL is equal to DG, and DA, DB, parts of them, are equal; therefore the remainder AL is equal to the remainder ^f BG: But it has been shewn, that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the same are equal to one another; therefore the straight line AL is equal to BC. Wherefore from the given point A a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

f 3. Ax.

PROP. III. PROB.

FROM the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.



a 2. 1.

b 3. Post.

From the point A draw ^a the straight line AD equal to C; and from the centre A, and at the distance AD, describe ^b the circle DEF; and because A is the centre of the circle DEF, AE shall be equal to AD; but the straight line C is likewise equal to AD; whence AE and C are each of them equal to AD; wherefore the straight line AE is equal to ^c C, and from AB, the greater of two straight lines, a part AE has been cut off equal to C the less. Which was to be done.

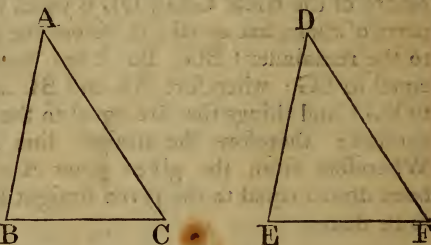
c 1. Ax.

PROP.

PROP. IV. THEOREM.

IF two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another; their bases, or *third sides*, shall be equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.

Let ABC , DEF be two triangles which have the two sides AB , AC equal to the two sides DE , DF , each to each, viz. AB to DE , and AC to DF ; and the angle BAC equal to the angle EDF , the base BC shall be equal to the base EF ; and the triangle ABC to the triangle DEF ; and the other angles, to B



C which the equal sides are opposite, shall be equal, each to each, viz. the angle ABC to the angle DEF , and the angle ACB to DFE .

For, if the triangle ABC be applied to the triangle DEF , so that the point A may be on D , and the straight line AB upon DE ; the point B shall coincide with the point E , because AB is equal to DE ; and AB coinciding with DE , AC shall coincide with DF , because the angle BAC is equal to the angle EDF ; wherefore also the point C shall coincide with the point F , because AC is equal to DF : But the point B coincides with the point E ; wherefore the base BC shall coincide with the base EF ^{a cor. def. 3.} and shall be equal to it. Therefore also the whole triangle ABC shall coincide with the whole triangle DEF , and be equal to it; and the remaining angles of the one shall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF , and the angle ACB to the angle DFE . Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, their bases shall be equal, and the

the triangles shall be equal, and their other angles, to which the equal sides are opposite, shall be equal, each to each. Which was to be demonstrated. Book I.

PROP. V. THEOR.

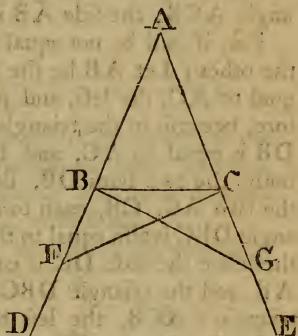
THE angles at the base of an Isosceles triangle are equal to one another; and, if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB, AC be produced to D and E, the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater cut off AG equal ^a to AF, the less, and join FC, GB. a 3. 1.

Because AF is equal to AG, and AB to AC, the two sides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG

common to the two triangles, AFC, AGB; therefore the base FC is equal ^b to the base GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal ^b to the remaining angles of the other, each to each, to which the equal sides are opposite, viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And



because the whole AF is equal to the whole AG, and the part AB to the part AC; the remainder BF shall be equal ^c to the remainder CG; and FC was proved to be equal ^{c 3. Ax.} to GB, therefore the two sides BF, FC are equal to the two CG, GB, each to each; but the angle BFC is equal to the angle CGB; wherefore the triangles BFC, CGB are equal ^b, and their remaining angles are equal, to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. Now, since it has been demonstrated, that the whole angle ABG is equal to the whole whole

Book I.

whole ACF , and the part CBG to the part BCF , the remaining angle ABC is therefore equal to the remaining angle ACB , which are the angles at the base of the triangle ABC : And it has also been proved that the angle FBC is equal to the angle GCB , which are the angles upon the other side of the base. Therefore the angles at the base, &c. **Q. E. D.**

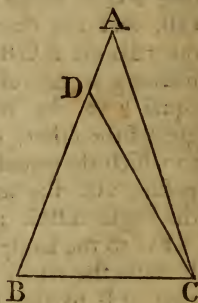
COROLLARY. Hence every equilateral triangle is also equiangular.

PROP. VI. THEOR.

IF two angles of a triangle be equal to one another, the sides which subtend, or are opposite to, those angles, shall also be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB ; the side AB is also equal to the side AC .

For, if AB be not equal to AC , one of them is greater than the other: Let AB be the greater, and from it cut ^a off DB equal to AC , the less, and join DC ; therefore, because in the triangles DBC , ACB , DB is equal to AC , and BC common to both, the two sides DB , BC are equal to the two AC , CB , each to each; but the angle DBC is also equal to the angle ACB ; therefore the base DC is equal to the base AB , and the triangle DBC is equal to the triangle ^b ACB , the less to the greater; which is absurd. Therefore AB is not unequal to AC , that is, it is equal to it. Wherefore, if two angles, &c. **Q. E. D.**



COR. Hence every equiangular triangle is also equilateral.

PROP.

PROP. VII. THEOR.

Book I.

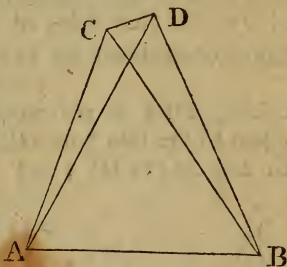
See N.

UPON the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity, equal to one another.

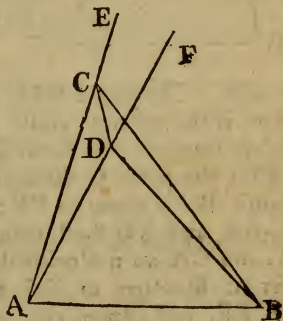
If it be possible, let there be two triangles ACB , ADB , upon the same base AB , and upon the same side of it, which have their sides CA , DA , terminated in A equal to one another, and likewise their sides CB , DB , terminated in B , equal to one another.

Join CD ; then, in the case in which the vertex of each of the triangles is without the other triangle, because AC is equal to AD , the angle ACD is equal to the angle ADC : But the angle ACD is greater than the angle BCD ; therefore the angle ADC is greater also than BCD ; much more then is the angle BDC greater than the angle BCD . Again, because CB is equal to DB , the angle BDC is equal to the angle BCD ; but it has been demonstrated to be greater than it; which is impossible.

But if one of the vertices, as D , be within the other triangle ACB ; produce AC , AD to E , F ; therefore, because AC is equal to AD in the triangle ACD , the angles ECD , FDC upon the other side of the base CD are equal to one another, but the angle ECD is greater than the angle BCD ; wherefore the angle FDC is likewise greater than BCD ; much more then is the angle BDC greater than the angle BCD . Again, because CB is equal to DB , the angle BDC is equal to the angle BCD ; but BDC has been



a 5. 1.



been.

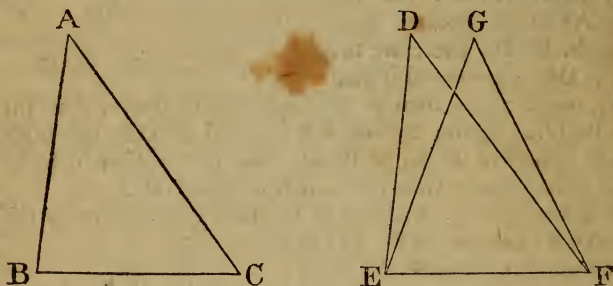
Book I. been proved to be greater than the same BCD; which is impossible. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

Therefore, upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity. Q. E. D.

PROP. VIII. THEOR.

IF two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides of the other.

Let ABC, DEF be two triangles having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF; and also the base BC equal to the



base EF. The angle BAC is equal to the angle EDF.

For, if the triangle ABC be applied to the triangle DEF, so that the point B be on E, and the straight line BC upon EF; the point C shall also coincide with the point F, because BC is equal to EF: therefore BC coinciding with EF, BA and AC shall coincide with ED and DF; for, if BA, and CA do not coincide with ED, and FD, but have a different situation as EG and FG; then, upon the same base EF, and upon the same side of it, there can be two triangles EDF, EGF, that have their sides which are

are terminated in one extremity of the base equal to one Book I.
 another, and likewise their sides terminated in the other
 extremity: But this is impossible ^a; therefore, if the base BC a 7. 1.
 coincides with the base EF, the sides BA, AC cannot but
 coincide with the sides ED, DF; wherefore likewise the
 angle BAC coincides with the angle EDF, and is equal ^b to b 8. Ax.
 it. Therefore if two triangles, &c. Q. E. D.

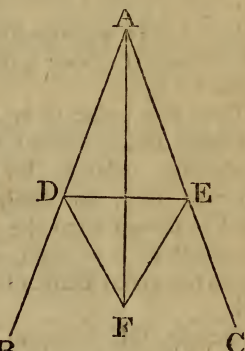
PROP. IX. PROB.

TO bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to bisect it.

Take any point D in AB, and from AC cut ^a off AE a 3. 1.
 equal to AD; join DE, and upon it
 describe ^b an equilateral triangle b 1. 1.
 DEF; then join AF; the straight
 line AF bisects the angle BAC.

Because AD is equal to AE, and
 AF is common to the two triangles
 DAF, EAF; the two sides DA,
 AF, are equal to the two sides EA,
 AF, each to each; but the base DF
 is also equal to the base EF; there-
 fore the angle DAF is equal ^c to the c 8. 1.
 angle EAF; wherefore the given
 rectilineal angle BAC is bisected by B
 the straight line AF. Which was to be done.



PROP. X. PROB.

TO bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line; it is required to divide it into two equal parts.

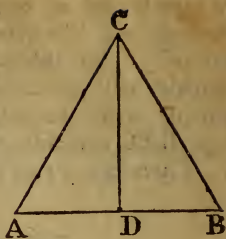
Describe ^a upon it an equilateral triangle ABC, and bisect a 1. 1.
^b the angle ACB by the straight line CD. AB is cut into b 9. 1.
 two equal parts in the point D.

Because

Book I.

c 4. I.

Because AC is equal to CB, and CD common to the two triangles ACD, BCD : the two sides AC, CD are equal to the two BC, CD, each to each ; but the angle ACD is also equal to the angle BCD ; therefore the base AD is equal to the base ^c DB, and the straight line AB is divided into two equal parts in the point D. Which was to be done.



2.E.3

PROP. XI. PROB.

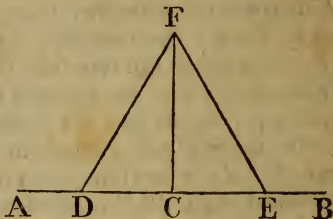
TO draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be a given straight line, and C a point given in it ; it is required to draw a straight line from the point C at right angles to AB.

a 3. I.

b 1. I.

Take any point D in AC, and ^a make CE equal to CD, and upon DE describe ^b the equilateral triangle DFE, and join FC ; the straight line FC, drawn from the given point C, is at right angles to the given straight line AB.



c 8. I.

d 7. def.

Because DC is equal to CE, and FC common to the two triangles DCF, ECF, the two sides DC, CF, are equal to the two EC, CF, each to each ; but the base DF is also equal to the base EF ; therefore the angle DCF is equal ^c to the angle ECF ; and they are adjacent angles. But, when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called a right ^d angle ; therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given straight line AB, FC has been drawn at right angles to AB. Which was to be done.

PROP.

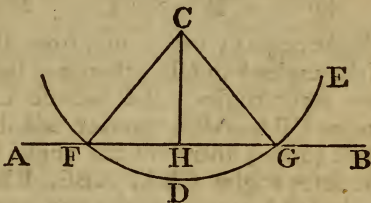
PROP. XII. PROB.

Book I.

TO draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a straight line perpendicular to AB from the point C .

Take any point D upon the other side of AB , and from the centre C , at the distance CD , describe ^a the circle EGF meeting AB in F, G ; and bisect ^b FG in H , and join CF, CH, CG ; the straight line CH , drawn from the given point C , is perpendicular to the given straight line AB .



a 3. Post.

b 10. 1.

Because FH is equal to HG , and HC common to the two triangles FHC, GHC , the two sides FH, HC are equal to the two GH, HC , each to each; now the base CF is also equal ^c to the base CG ; therefore the angle CHF is equal ^d to the angle CHG ; and they are adjacent angles; but when a straight line standing on a straight line makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it; therefore from the given point C a perpendicular CH has been drawn to the given straight line AB . Which was to be done.

c 11. Def.

d 8. 1.

PROP. XIII. THEOR.

THE angles which one straight line makes with another upon the one side of it, are either two right angles, or are together equal to two right angles.

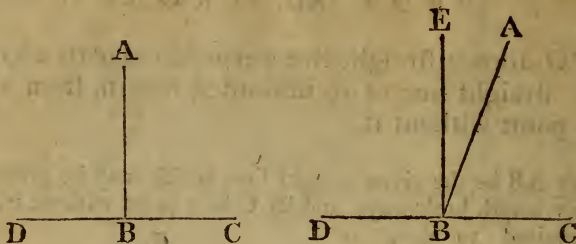
Let the straight line AB make with CD , upon one side of it, the angles CBA, ABD ; these are either two right angles, or are together equal to two right angles.

C

For

Book I.

For, if the angle CBA be equal to ABD, each of them is



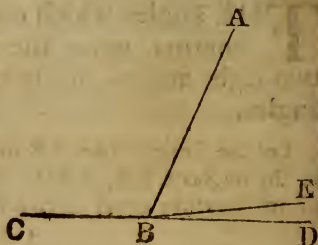
- a def. 7. a right angle^a; but, if not, from the point B draw BE at
 b II. 1. right angles^b to CD; therefore the angles CBE, EBD are
 two right angles; and because CBE is equal to the two
 angles CBA, ABE together, add the angle EBD to each of
 c 2. Ax. these equals; therefore the angles CBE, EBD are equal^c to
 the three angles CBA, ABE, EBD. Again, because the
 angle DBA is equal to the two angles DBE, EBA, add to
 these equals the angle ABC; therefore the angles DBA,
 ABC are equal to the three angles DBE, EBA, ABC; but
 the angles CBE, EBD have been demonstrated to be equal to
 the same three angles; and things that are equal to the same
 d 1. Ax. are equal^d to one another; therefore the angles CBE, EBD
 are equal to the angles DBA, ABC; but CBE, EBD are two
 right angles; therefore DBA, ABC are together equal to
 two right angles. Wherefore, when a straight line, &c.
 Q. E. D.

PROP. XIV. THEOR.

IF, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB, make the adjacent angles ABC, ABD equal together to two right angles. BD is in the same straight line with CB.

For, if BD be not in the



let BE be in the same straight line with it; therefore, because the straight line AB makes angles with the straight line CBE, upon one side of it, the angles ABC, ABE are together equal ^a to two right angles; but the angles ABC, ABD are likewise together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: Take away the common angle ABC, and the remaining angle ABE is equal ^b to the remaining angle ABD, the less to the greater, which is impossible; therefore BE is not in the same straight line with BC. And, in like manner, it may be demonstrated, that no other can be in the same straight line with it but BD, which therefore is in the same straight line with CB. Wherefore, if at a point, &c. Q. E. D.

PROP. XV. THEOR.

IF two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E; the angle AEC shall be equal to the angle DEB, and CEB to AED.

For the angles CEA, AED, which the straight line AE makes with the straight line CD, are together equal ^a to two right angles: and the angles AED, DEB, which the straight line DE makes with the straight line AB, are also together equal ^a to two right angles; therefore the two angles CEA, AED are equal to the two AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal ^b to the remaining angle DEB. In the same manner it can be demonstrated that the angles CEB, AED are equal. Therefore, if two straight lines, &c. Q. E. D.

COR. 1. From this it is manifest, that, if two straight lines cut one another, the angles which they make at the point of their intersection, are together equal to four right angles.

COR. 2. And hence, all the angles made by any number of lines meeting in one point, are together equal to four right angles.

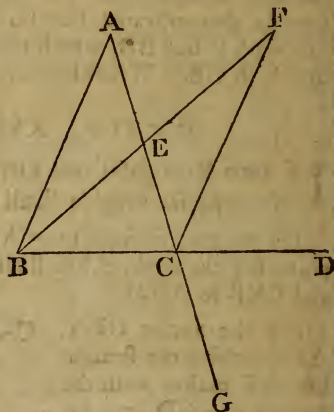
PROP. XVI. THEOR.

IF one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let its sides BC be produced to D , the exterior angle ACD is greater than either of the interior opposite angles CBA , BAC .

a 10. I.

Bisect AC in E , join BE and produce it to F , and make EF equal to BE ; join also FC , and produce AC to G .



b 15. I.

Because AE is equal to EC , and BE to EF ; AE , EB are equal to CE , EF , each to each; and the angle AEB is equal b to the angle CEF , because they are opposite vertical angles; therefore the base

c 4. I.

AB is equal c to the base CF , and the triangle AEB to the triangle CEF , and the remaining angles to the remaining

d 15. I.

angles, each to each, to which the equal sides are opposite: wherefore the angle BAE is equal to the angle ECF ; but the angle ECD is greater than the angle ECF ; therefore the angle ECD , that is ACD , is greater than BAE : In the same manner, if the side BC be bisected, it may be demonstrated that the angle BCG , that is d , the angle ACD , is greater than the angle ABC . Therefore, if one side, &c. Q. E. D.

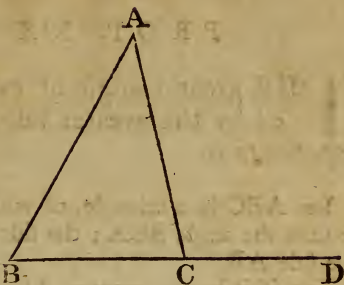
PROP. XVII. THEOR.

ANY two angles of a triangle are together less than two right angles.

Let

Let ABC be any triangle; any two of its angles together are less than two right angles.

Produce BC to D ; and because ACD is the exterior angle of the triangle ABC , ACD is greater^a than the interior and opposite angle ABC ; to each of these add the angle ACB ; therefore the angles ACD , ACB are greater than the angles ABC , ACB ; but ACD , ACB are together equal^b to two right angles; therefore the angles ABC , BCA are less than two right angles. In like manner, it may be demonstrated, that BAC , ACB , as also CAB , ABC , are less than two right angles. Therefore, any two angles, &c. Q. E. D.



a 16. I.

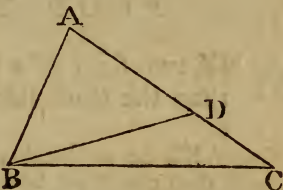
b 13. 2.

PROP. XVIII. THEOR.

THE greater side of every triangle has the greater angle opposite to it.

Let ABC be a triangle, of which the side AC is greater than the side AB ; the angle ABC is also greater than the angle BCA .

Because AC is greater than AB , make^a AD equal to AB , and join BD ; and because ADB is the exterior angle of the triangle BDC , it is greater^b than the interior and opposite angle DCB ; but ADB is equal^c to ABD , because the side AB is equal to the side AD ; therefore the angle ABD is likewise greater than the angle ACB ; wherefore much more is the angle ABC greater than ACB . Therefore the greater side, &c. Q. E. D.



a 3. I.

b 16. I.

c 5. I.

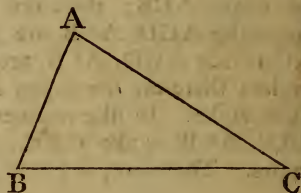
Book I.

PROP. XIX. THEOR.

THE greater angle of every triangle is subtended by the greater side, or *has the greater side opposite to it.*

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA ; the side AC is likewise greater than the side AB .

For, if it be not greater, AC must either be equal to AB , or less than it; it is not equal, because then the angle ABC would be equal ^a to the angle ACB ; but it is not; therefore AC is not equal to AB ; neither is it less; because then the angle



ABC would be less ^b than the angle ACB ; but it is not; therefore the side AC is not less than AB ; and it has been shewn that it is not equal to AB ; therefore AC is greater than AB . Wherefore the greater angle, &c. Q. E. D.

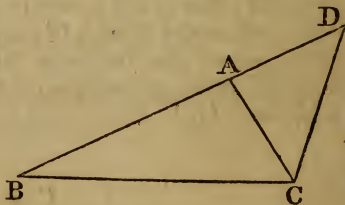
PROP. XX. THEOR.

ANY two sides of a triangle are together greater than the third side.

Let ABC be a triangle; any two sides of it together are greater than the third side, *viz.* the sides BA , AC greater than the side BC ; and AB , BC greater than AC ; and BC , CA greater than AB .

Produce BA to the point D , and make ^a AD equal to AC ; and join DC .

Because DA is equal to AC , the angle ADC is likewise equal ^b to ACD ; but the angle BCD is greater than the angle ACD ; there-



fore the angle BCD is greater than the angle ADC ; and be-
cause

cause the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater side is opposite to the greater angle; therefore the side DB is greater than the side BC; but DB is equal to BA and AC; therefore the sides BA, AC are greater than BC. In the same manner it may be demonstrated, that the sides AB, BC are greater than CA, and BC, CA greater than AB. Therefore any two sides, &c.
Q. E. D.

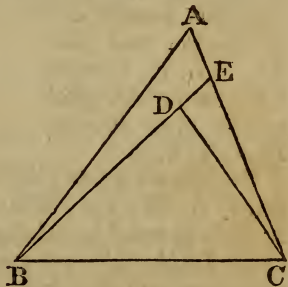
PROP. XXI. THEOR.

IF from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle, these two lines shall be less than the other two sides of the triangle, but shall contain a greater angle.

See N.

Let the two straight lines BD, CD be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it; BD and DC are less than the other two sides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and because two sides of a triangle are greater than the third side, the two sides BA, AE of the triangle ABE are greater than BE. To each of these add EC; therefore the sides BA, AC are greater than BE, EC: Again, because the two sides CE, ED of the triangle CED are greater than CD, add DB to each of these; therefore the sides CE, EB are greater than CD, DB; but it has been shewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again, because the exterior angle of a triangle is greater than the interior and opposite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the same reason, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonstrated that the angle

Book I.

$\angle BDC$ is greater than the angle $\angle CEB$; much more then is the angle $\angle BDC$ greater than the angle $\angle BAC$. Therefore, if from the ends of, &c. Q. E. D.

PROP. XXII. PROB.

TO make a triangle of which the sides shall be equal to three given straight lines; but any two whatever of these lines must be greater than the third^a.

a 2D. I.

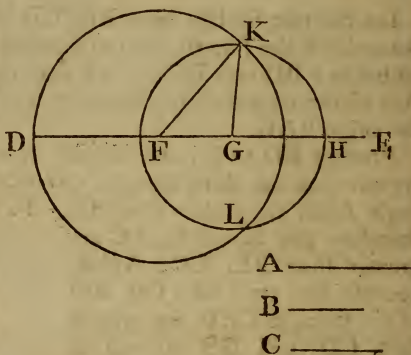
Let A, B, C be the three given straight lines, of which any two whatever are greater than the third. viz. A and B greater than C ; A and C greater than B ; and B and C than A . It is required to make a triangle of which the sides shall be equal to A, B, C , each to each.

Take a straight line DE terminated at the point D , but unlimited towards E ,

a 3. I.

and make^a DF equal to A , FG to B , and GH equal to C ; and from the centre F , at the distance FD , describe^b the circle DKL ; and from the centre G , at the distance GH , describe^b another circle HLK ; and join KF, KG ; the triangle KFG has its sides equal to the three straight lines, A, B, C .

b 3. Post.



Because the point F is the centre of the circle DKL , FD is equal^c to FK ; but FD is equal to the straight line A ; therefore FK is equal to A : Again, because G is the centre of the circle LKH , GH is equal^c to GK ; but GH is equal to C ; therefore also GK is equal to C ; and FG is equal to B ; therefore the three straight lines KF, FG, GK , are equal to the three A, B, C : And therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines, A, B, C . Which was to be done.

c 11. Def.

PROP.

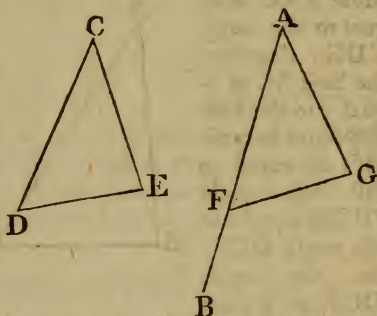
PROP. XXIII. PROB.

Book I.

AT a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given straight line AB , that shall be equal to the given rectilineal angle DCE .

Take in CD , CE any points D , E , and join DE ; and make^a the triangle AFG , the sides of which shall be equal to the three straight lines CD , DE , CE , so that CD be equal to AF , CE to AG , and DE to FG ; and because DC , CE are equal to FA , AG , each to each, and the base DE to the base FG ; the angle DCE is equal^b to the angle FAG . Therefore, at the given point A in the given straight line AB , the angle FAG is made equal to the given rectilineal angle DCE . Which was to be done.



a 22. 1.

b 8. 1.

PROP. XXIV. THEOR.

IF two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let ABC , DEF be two triangles which have the two sides AB , AC equal to the two DE , DF , each to each, viz. AB equal

Book I.

equal to DE , and AC to DF ; but the angle BAC greater than the angle EDF ; the base BC is also greater than the base EF .

Of the two sides DE , DF , let DE be the side which is not greater than the other, and at the point D , in the straight line

a 23. 1. DE , make ^a the angle EDG equal to the angle BAC ; and
b 3. 1. make DG equal ^b to AC or DF , and join EG , GF .

Because AB is equal to DE , and AC to DG , the two sides BA , AC are equal to the two ED , DG , each to each, and the angle BAC is e-

qual to the angle EDG ; therefore the base BC is e-

qual ^c to the base EG ; and because

DG is equal to

DF , the angle

DFG is equal ^d to the angle DGF ; but the angle

DGF is greater

than the angle EGF ; therefore the angle DFG is greater

than EGF ; and much more is the angle EFG greater than the angle EGF ; and because the angle EFG of the triangle

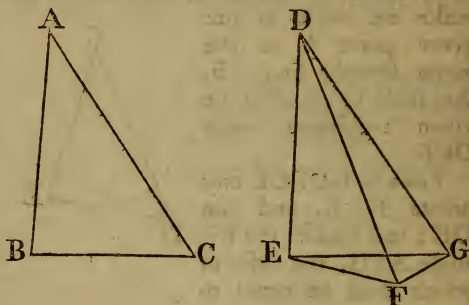
EFG is greater than its angle EGF , and that the greater ^e

side is opposite to the greater angle; the side EG is therefore

greater than the side EF ; but EG is equal to BC ; and there-

fore also BC is greater than EF . Therefore, if two triangles,

&c. Q. E. D.



PROP. XXV. THEOR.

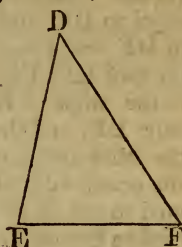
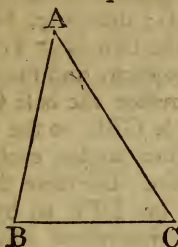
IF two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

Let ABC , DEF be two triangles which have the two sides AB , AC equal to the two sides DE , DF , each to each, viz.

AB

AB equal to DE, and AC to DF; but the base CB is greater than the base EF; the angle BAC is likewise greater than the angle EDF.

For, if it be not greater, it must either be equal to it, or less; but the angle BAC is not equal to the angle EDF, because then the base BC would be equal ^a to EF; but it is not; therefore the angle BAC is not equal to the angle EDF; neither is it less; because then the base BC would be less ^b than the base EF; but it is not; therefore the angle BAC is not less than the angle EDF; and it was shewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, &c. Q. E. D.



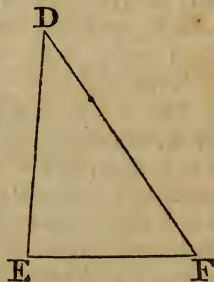
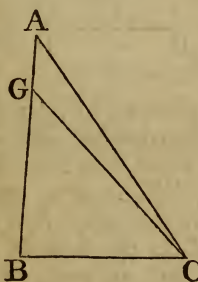
a 4. I.

b 24. I.

PROP. XXVI. THEOR.

IF two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, viz. either the sides adjacent to the equal angles, or the sides opposite to the equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; also one side equal to one side; and first let those sides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF; the

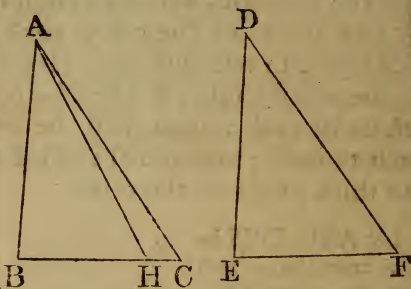


Book I

the other sides shall be equal, each to each, viz. AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.

For, if AB be not equal to DE, one of them must be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore, because BG is equal to DE, and BC to EF, the two sides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the base GC is equal^a to the base DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypothesis, equal to the angle BCA; wherefore also the angle BCG is equal to the angle BCA, the less to the greater, which is impossible; therefore AB is not unequal to DE, that, is, it is equal to it; and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; the base therefore AC is equal^a to the base DF, and the third angle BAC to the third angle EDF.

Next, let the sides which are opposite to equal angles in each triangle be equal to one another, viz. AB to DE; likewise in this case, the other sides shall be equal, AC to DF, and BC to EF; and also the third angle BAC to the third EDF.



For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH; and because BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DEF, and the other angles shall be equal, each to each, to which the equal sides are opposite; therefore the angle BHA is equal to the angle EFD;

EFD; but EFD is equal to the angle BCA; therefore also the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and opposite angle BCA; which is impossible^b; wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC, are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, &c. Q. E. D.

PROP. XXVII. THEOR.

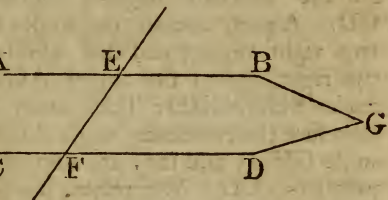
IF a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD make the alternate angles AEF, EFD equal to one another; AB is parallel to CD

For, if it be not parallel, AB and CD being produced shall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater^a than the interior and opposite angle EFG; but it is

also equal to it, which is impossible; therefore, AB and CD being produced, do not meet towards B, D. In like manner it may be demonstrated that they do not meet towards A, C; but those

straight lines which meet neither way, though produced ever so far, are parallel^b to one another. AB therefore is parallel to CD. Wherefore, if a straight line, &c. Q. E. D.



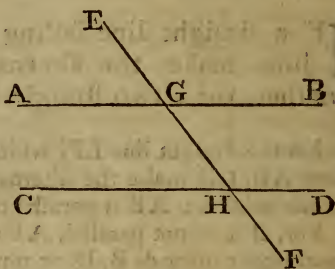
PROB

Book I.

PROP. XXVIII. THEOR.

IF a straight line falling upon two other straight lines makes the exterior angle equal to the interior and opposite upon the same side of the line; or makes the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side; or make the interior angles on the same side BGH, GHD together equal to two right angles; AB is parallel to CD.



a 15. 1.

b 27. 1.

c By Hyp.

d 13. 1.

Because the angle EGB is equal to the angle GHD, and the angle EGB equal a to the angle AGH, the angle AGH is equal to the angle GHD; and they are the alternate angles; therefore AB is parallel b to CD. Again, because the angles BGH, GHD are equal c to two right angles; and that AGH, BGH, are also equal d to two right angles; the angles AGH, BGH are equal to the angles BGH, GHD: Take away the common angle BGH; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD. Wherefore, if a straight line, &c. Q. E. D.

PROP. XXIX. THEOR.

See N.

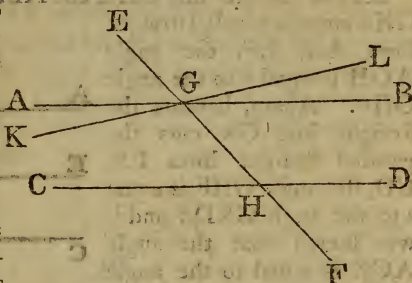
IF a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let

Let the straight line EF fall upon the parallel straight lines AB , CD ; the alternate angles AGH , GHD are equal to one another; and the exterior angle EGB is equal to the interior and opposite, upon the same side, GHD ; and the two interior angles BGH , GHD upon the same side are together equal to two right angles.

For if AGH be not equal to GHD , let KG be drawn making the angle KGH equal to GHD , and produce KG to L ; then KL

will be parallel to CD ^a; but AB is also parallel to CD ; therefore two straight lines are drawn through the same point G , parallel to CD , and yet not coinciding with one another, which is impossible^b. The angles



a 27. 5.

b 11. Ax.

AGH , GHD therefore are not unequal, that is, they are equal to one another. Now, the angle EGB is equal to AGH ^c; and AGH is proved to be equal to GHD ; therefore EGB is likewise equal to GHD ; add to each of these the angle BGH ; therefore the angles EGB , BGH are equal to the angles BGH , GHD ; but EGB , BGH are equal^d to two right angles; therefore also BGH , GHD are equal to two right angles. Wherefore, if a straight line, &c. Q. E. D.

c 15. 1.

d 13. 1.

COR. If two lines KL and CD make, with EF , the two angles KGH , GHC together less than two right angles, KG and GH will meet on the side of EF on which the two angles are that are less than two right angles.

For, if not, KL and CD are either parallel, or they meet on the other side of EF ; but they are not parallel; for the angles KGH , GHC would then be equal to two right angles. Neither do they meet toward the points L and D ; for the angles LGH , GHD would then be two angles of a triangle, and less than two right angles; but this is impossible; for the four angles KGH , HGL , CHG , GHD are together equal to four right angles^d, of which the two KGH , CHG are by supposition less than two right angles; therefore the other two, HGL , GHD are greater than two right angles. Therefore, since KL and CD are not parallel, and do not meet towards L and D , they will meet if produced towards K and C .

P R O P.

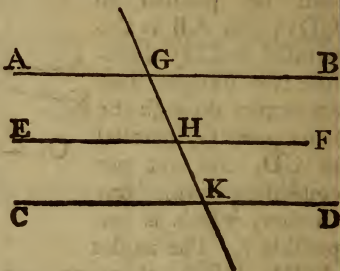
Book I.

PROP. XXX. THEOR.

STRAIGHT lines which are parallel to the same straight line are parallel to one another.

Let AB , CD be each of them parallel to EF ; AB is also parallel to CD .

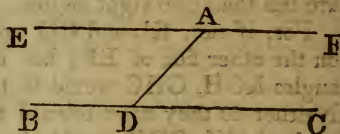
Let the straight line GHK cut AB , EF , CD ; and because GHK cuts the parallel straight lines AB , EF , the angle AGH is equal ^a to the angle GHE . Again, because the straight line GK cuts the parallel straight lines EF , CD , the angle GHE is equal ^a to the angle GKD ; and it was shewn that the angle AGK is equal to the angle GHE ; therefore also AGK is equal to GKD ; and they are alternate angles; therefore AB is parallel ^b to CD . Wherefore straight lines, &c. $Q.E.D.$



PROP. XXXI. PROB.

TO draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw a straight line through the point A , parallel to the straight line BC .



In BC take any point D , and join AD ; and at the point A , in the straight line AD make ^a the angle DAE equal to the angle ADC ; and produce the straight line EA to F .

Because the straight line AD , which meets the two straight lines BC , EF , makes the alternate angles EAD , ADC equal to one another, EF is parallel ^b to BC . Therefore the straight

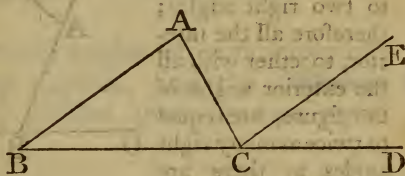
line EAF is drawn through the given point A parallel to the given straight line BC. Which was to be done. Book I.

PROP. XXXII. THEOR.

IF a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

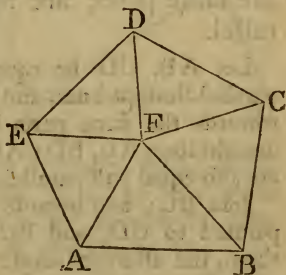
Let ABC be a triangle, and let one of its sides BG be produced to D; the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC and the three interior angles of the triangle; viz. ABC, BCA, CAB, are together equal to two right angles.

Through the point C draw CE parallel^a to the straight line AB; and because AB is parallel to CE and AC meets them, the alternate angles BAC, ACE are equal^b. Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC; but the angle ACE was shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC; to these angles add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal^c to two right angles; therefore also the angles CBA, BAC, ACB are equal to two right angles. Wherefore, if a side of a triangle, &c. Q. E. D.



COR. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For any rectilineal figure BCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the preceding proposition, all



Book I,

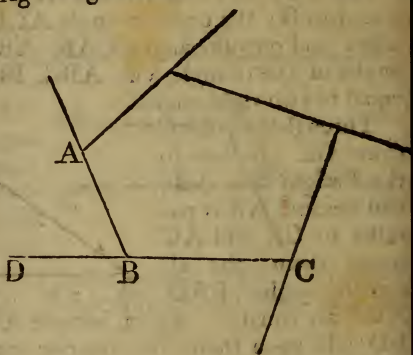
a 2. Cor.
15. 1.

all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles: that is ^a, together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

COR. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

b 13. 1.

Because every interior angle ABC, with its adjacent exterior ABD, is equal ^b to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure; that is by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.

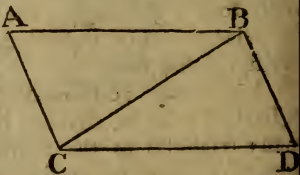


PROP. XXXIII. THEOR.

THE straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.

Let AB, CD be equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD; AC, BD are also equal and parallel.

Join BC; and because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal ^a; and be



a 29. 1.

Book I.

Because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; therefore the base AC is equal^b to the base BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles^b, each to each, to which the equal sides are opposite; therefore the angle ACB is equal to the angle CBD; and because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel^c to BD; and it was shewn to be equal to it. Therefore, straight lines, &c. Q. E. D. c 27. 1.

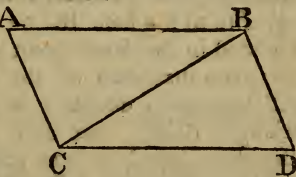
PROP. XXXIV. THEOR.

THE opposite sides and angles of parallelograms are equal to one another, and the ~~diameter~~ *diagonal* bisects them, that is, divides them in two equal parts.

N. B. A parallelogram is a four sided figure, of which the opposite sides are parallel; and the ~~diameter~~ *diagonal* is the straight line joining two of its opposite angles.

Let ACDB be a parallelogram, of which BC is a diameter; the opposite sides and angles of the figure are equal to one another; and the diameter BC bisects it.

Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal^a to one another; and because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal^a to one another; wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other, each to each, and one side BC common to the two triangles, which is adjacent to their equal angles;



a 29. 1.

D 2

B I angles; therefore their other sides shall be equal, each each, and the third angle of the one to the third angle of the other *b*, viz. the side AB to the side CD, and AC to BD, and the angle BAC equal to the angle BDC: And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD: And the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of parallelograms are equal to one another also, their diameter bisects them; for AB being equal to CD and BC common, the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC is equal to the angle BCD; therefore the triangle ABC is equal *c* to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts. Q. E. D.

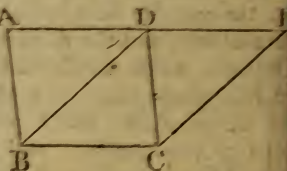
PROP. XXXV. THEOR.

PARALLELOGRAMS upon the same base and between the same parallels, are equal to one another.

See the 2d and 3d figures.

Let the parallelograms ABCD, EBCF be upon the same base BC, and between the same parallels AF, BC; the parallelogram ABCD shall be equal to the parallelogram EBCF.

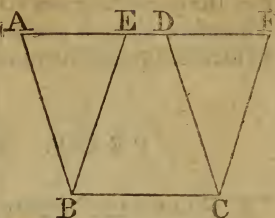
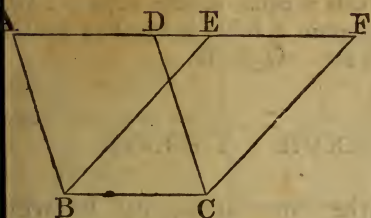
If the sides AD, DF of the parallelograms ABCD, EBCF opposite to the base BC be terminated in the same point D; it is plain that each of the parallelograms is double *a* of the triangle BDC; and they are therefore equal to one another.



But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point; then, because ABCD is a parallelogram, AD is equal *b* to BC; for the same reason EF is equal to BC; wherefore AD is equal *b* to EF; and DE is common; therefore the whole, or the remainder, AE is equal *c* to the whole, *AE*.

the remainder DF ; AB also is equal to DC ; and the two

Book I.

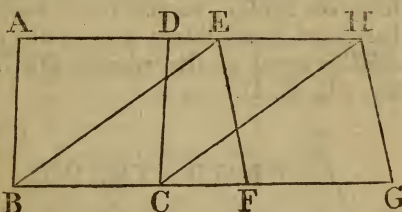


AB , AB are therefore equal to the two FD , DC , each to each; and the exterior angle FDC is equal to the interior angle EDC ; therefore the base EB is equal to the base FC , and the triangle EAB equal to the triangle FDC ; take the triangle FDC from the trapezium $ABCF$, and from the same trapezium take the triangle EAB ; the remainders therefore are equal, that is, the parallelogram $ABCD$ is equal to the parallelogram $EBCF$. Therefore, parallelograms upon the same base, &c. Q. E. D.

PROP. XXXVI. THEOR.

PARALLELOGRAMS upon equal bases, and between the same parallels, are equal to one another.

Let $ABCD$, $EFGH$ be parallelograms upon equal bases BC , FG , and between the same parallels AH , BG ; the parallelogram $ABCD$ is equal to $EFGH$.



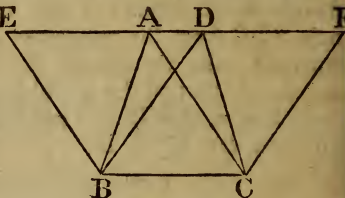
Join BE , CH ; and because BC is equal to FG , and FG to EH , BC is equal to EH ; and they are parallels, and joined towards the same parts by the straight lines BE , CH : But straight lines which are terminated in equal and parallel straight lines towards the same parts, are themselves equal and parallel; therefore BE , CH are both equal and parallel, and $EBCH$ is a parallelogram; and it is equal to $ABCD$, because it is upon the same base BC , and

Book I. and between the same parallels BC, AD : For the like reason, the parallelogram $EFGH$ is equal to the same $EBCH$. Therefore also the parallelogram $ABCD$ is equal to $EFGH$. Wherefore parallelograms, &c. Q. E. D.

PROP. XXXVII. THEOR.

TRIANGLES upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC and between the same parallels AD, BC : The triangle ABC is equal to the triangle DBC .



Produce AD both ways to the points E, F , and through B draw a BE parallel to CA ; and through C draw CF parallel to BD : Therefore, each of the figures $EBCA, DBCF$ is a parallelogram; and $EBCA$ is equal ^b to $DBCF$, because they are upon the same base BC , and between the same parallels BC, EF ; and the triangle ABC is the half of the parallelogram $EBCA$, because the diameter AB bisects ^c it; and the triangle DBC is the half of the parallelogram $DBCF$, because the diameter DC bisects it: But the halves of equal things are equal ^d; therefore the triangle ABC is equal to the triangle DBC . Wherefore triangles &c. Q. E. D.

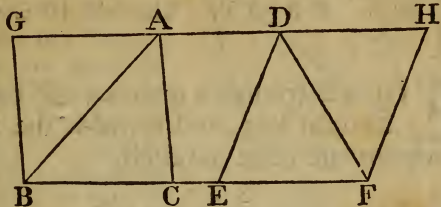
PROP. XXXVIII. THEOR.

TRIANGLES upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bases BC, EF , and between the same parallels BF, AD : The triangle ABC is equal to the triangle DEF .

Produce

Produce AD both ways to the points G, H, and through G draw BG parallel ^a to CA, and through F draw FH parallel to ED: Then each of the figures GBCA, DEFH is a parallelogram; and they are equal to ^b one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH; and the triangle ABC is the half ^c of the parallelogram GBCA, because the diameter AB bisects it; and the triangle DEF is the half ^c of the parallelogram DEFH, because the diameter DF bisects it: But the halves of equal things are equal ^d; therefore the triangle ABC is equal to the triangle DEF. Wherefore triangles, &c. Q. E. D.



b 36. 1.

c 34. 1.

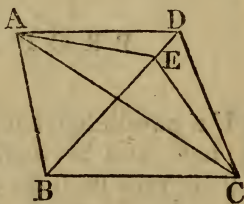
d 7. Ax.

PROP. XXXIX. THEOR.

EQUAL triangles upon the same base, and upon the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and upon the same side of it; they are between the same parallels.

Join AD; AD is parallel to BC; for, if it is not, through the point A draw ^a AE parallel to BC, and join EC: The angle ABC is equal ^b to the triangle EBC, because it is upon the same base BC, and between the same parallels BC, AE: But the angle ABC is equal to the triangle DCB; therefore also the triangle DCB is equal to the triangle EBC, the greater to the less, which is impossible: Therefore AE is not parallel to BC. In the same manner, it can be demonstrated that no other line but AD is parallel to BC; AD is therefore parallel to it. Wherefore equal triangles upon, &c. Q. E. D.



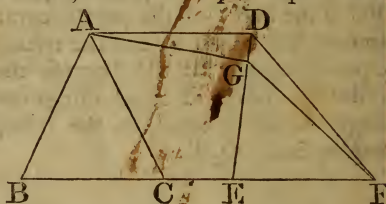
a 31. 1.

b 37. 1.

PROP. XL. THEOR.

EQUAL triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.

Let the equal triangles ABC , DEF be upon equal bases BC , EF , in the same straight line BF , and towards the same parts; they are between the same parallels.



Join AD ; AD is parallel to BC : For, if it is not, through A draw a AG parallel to BF , and join GF . The triangle ABC is equal b to the triangle GEF , because they are upon equal bases BC , EF , and between the same parallels BF , AG : But the triangle ABC is equal to the triangle DEF ; therefore also the triangle DEF is equal to the triangle GEF , the greater to the less, which is impossible: Therefore AG is not parallel to B BF : And in the same manner it can be demonstrated that there is no other parallel to it but AD ; AD is therefore parallel to BF . Wherefore equal triangles, &c. $Q. E. D.$

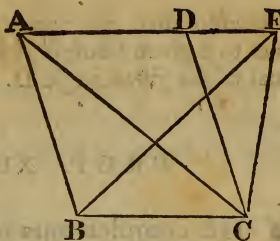
PROP. XLI. THEOR.

IF a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.

Let the parallelogram $ABCD$ and the triangle EBC be upon the same base BC , and between the same parallels BC , AE

a 37. I.

b 34. I.



AE; the parallelogram ABCD is double of the triangle EBC.

Join AC; then the triangle ABC is equal^a to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE. But the parallelogram ABCD is double^b of the triangle ABC, because the diameter AC divides it into two equal parts; wherefore ABCD is also double of the triangle EBC. Therefore, if a parallelogram, &c. Q. E. D.

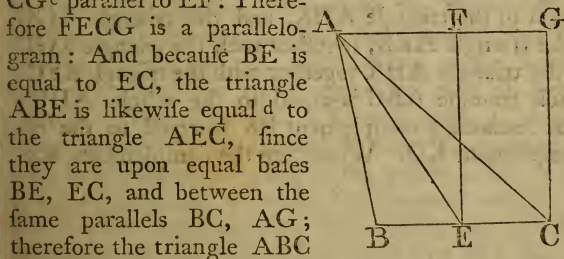
P R O P. XLII. P R O B.

TO describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

Bisect^a BC in E, join AE, and at the point E in the straight line EC make^b the angle CEF equal to D; and through A draw^c AG parallel to EC, and through C draw CG^c parallel to EF: There-

d 38. I.



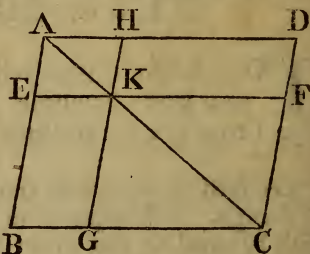
fore FECC is a parallelogram: And because BE is equal to EC, the triangle ABE is likewise equal^d to the triangle AEC, since they are upon equal bases BE, EC, and between the same parallels BC, AG; therefore the triangle ABC is double of the triangle AEC. And the parallelogram FECC is likewise double^e of the triangle AEC, because it is upon the same base, and between the same parallels: Therefore the parallelogram FECC is equal to the triangle ABC, and it has one of its angles CEF equal to the given angle D: Wherefore

Book I. Wherefore there has been described a parallelogram $FECG$ equal to a given triangle ABC , having one of its angles CEF equal to the given angle D . Which was to be done.

PROP. XLIII. THEOR.

THE complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC ; let EH, FG the parallelograms about AC , that is, thro' which AC passes, and BK, KD the other parallelograms which make up the whole figure $ABCD$, which are therefore called the complements: The complement BK is equal to the complement KD .



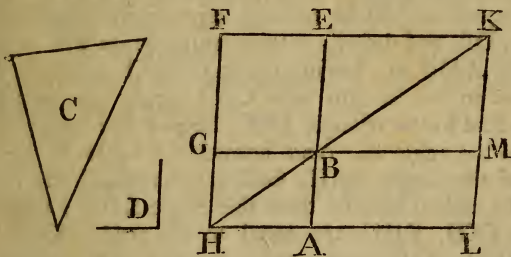
Because $ABCD$ is a parallelogram, and AC its diameter, the triangle ABC is equal ^a to the triangle ADC : And, because $EKHA$ is a parallelogram, the diameter of which is AK , the triangle AEK is equal to the triangle AHK : For the same reason, the triangle KGC is equal to the triangle KFC . Then, because the triangle AEK is equal to the triangle AHK , and the triangle KGC to KFC ; the triangle AEK , together with the triangle KGC is equal to the triangle AHK together with the triangle KFC : But the whole triangle ABC is equal to the whole ADC ; therefore the remaining complement BK is equal to the remaining complement KD . Wherefore the complements, &c.
Q. E. D.

PROP. XLIV. PROB.

TO a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the straight line AB a parallelogram equal to the triangle C , and having an angle equal to D .

Make a the parallelogram $BEFG$ equal to the triangle C , a 42. 1.



and having the angle EBG equal to the angle D , so that BE be in the same straight line with AB , and produce FG to H ; and through A draw b AH parallel to BG or EF , and join b 31. 1. HB . Then because the straight line HF falls upon the parallels AH , EF , the angles AHF , HFE , are together equal c 29. 1. to two right angles; wherefore the angles BHF , HFE are less than two right angles: But straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet if produced far enough: Therefore HB , FE shall meet, if produced; let them meet in K , and through K draw KL parallel to EA or FH , and produce HA , GB to the points L , M : Then $HLKF$ is a parallelogram, of which the diameter is HK , and AG , ME are the parallelograms about HK ; and LB , BF are the complements; therefore LB is equal d to BF : But BF is e d 43. 1. equal to the triangle C ; wherefore LB is equal to the triangle C ; and because the angle GBE is equal e e 15. 1. to the angle ABM , and likewise to the angle D ; the angle ABM is equal to the angle D : Therefore the parallelogram LB

Book I. LB is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D: Which was to be done.

PROP. XLV. PROB.

TO describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

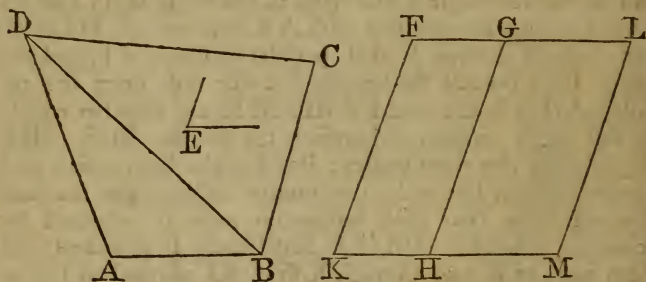
Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

a 42. I.

b 44. I.

Join DB, and describe ^a the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E; and to the straight line GH ^b apply the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E. And because the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add to each of these the angle KHG; therefore the angles FKH, KHG are equal to the angles KHG, GHM; but FKH, KHG are equal ^c to two right angles; there-

c 29. I.



d 14. I.

fore also KHG, GHM are equal to two right angles: and because at the point H in the straight line GH, the two straight lines KH, HM, upon the opposite sides of GH, make the adjacent angles equal to two right angles, KH is in the same straight line ^d with HM. And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal ^c; add to each of these the angle HGL; therefore the angles MHG, HGL, are equal to the angles HGF, HGL:

But

But the angles MHG, HGL are equal ^c to two right angles; ^{Book I.} wherefore also the angles HGF, HGL are equal to two right angles, and FG is therefore in the same straight line with GL. And because KF is parallel to HG, and HG to ML, KF is parallel ^e to ML; but KM, FL are parallels; wherefore KFLM is a parallelogram. And because the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM, the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM; therefore the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done. ^{e 30. I.}

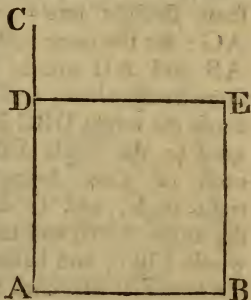
COR. From this it is manifest how to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, *viz.* by applying ^b to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle. ^{b 44. I.}

PROP. XLVI. PROB.

TO describe a square upon a given straight line.

Let AB be the given straight line; it is required to describe a square upon AB.

From the point A draw ^a AC at right angles to AB; and ^{a 11. I.} make ^b AD equal to AB, and through the point D draw ^{b 3. I.} DE parallel ^c to AB, and through B draw BE parallel to ^{c 31. I.} AD; therefore ADEB is a parallelogram: Whence AB is equal ^d to DE, and AD to BE: But ^{d 34. I.} BA is equal to AD; therefore the four straight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral: it is likewise rectangular; for the straight line AD meeting the parallels AB, DE, makes the angles BAD, ADE equal ^e to two right angles; but BAD is a right angle; therefore also ADE is a right angle; therefore also ADE is a right angle; now the opposite angles of parallelo-



Book I. **g**rams are equal **d**; therefore each of the opposite angles **ABE**, **BED** is a right angle; wherefore the figure **ADEB** is rectangular, and it has been demonstrated that it is equilateral; it is therefore a square, and it is described upon the given straight line **AB**: Which was to be done.

COR. Hence every parallelogram that has one right angle has all its angles right angles.

PROP. XLVII. THEOR.

IN any right angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

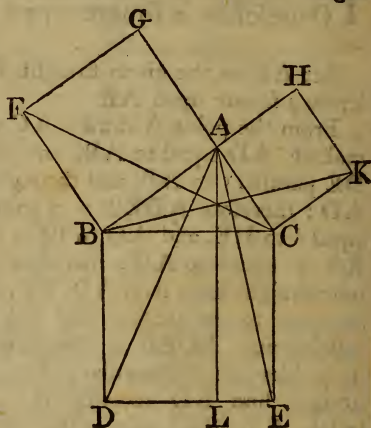
Let **ABC** be a right angled triangle having the right angle **BAC**; the square described upon the side **BC** is equal to the squares described upon **BA**, **AC**.

a 46. I. On **BC** describe ^a the square **BDEC**, and on **BA**, **AC** the
b 31. I. squares **GB**, **HC**; and through **A** draw ^b **AL** parallel to **BD**
 or **CE**, and join **AD**, **FC**; then, because each of the angles

c 30. def. **BAC**, **BAG** is a right
 angle ^c, the two straight
 lines **AC**, **AG** upon the
 opposite sides of **AB**,
 make with it at the point
A the adjacent angles e-
 qual to two right angles;

therefore **CA** is in the
d 14. I. same straight line ^d with
AG; for the same reason,
AB and **AH** are in the
 same straight line; and be-
 cause the angle **DBC** is e-
 qual to the angle **FBA**,
 each of them being a
 right angle, add to each

e 2. Ax. the angle **ABC**, and the whole angle **DBA** is equal ^e to the
 whole **FBC**; and because the two sides **AB**, **BD** are equal to
 the two **FB**, **BC**, each to each, and the angle **DBA** equal to
 the



the angle FBC ; therefore the base AD is equal ^f to the base FC , and the triangle ABD to the triangle FBC : Now the parallelogram BL is double ^g of the triangle ABD , because they are upon the same base BD , and between the same parallels, BD , AL ; and the square GB is double of the triangle FBC , because these also are upon the same base FB , and between the same parallels FB , GC . But the doubles of equals are equal ^h to one another; therefore the parallelogram BL is equal to the square GB : And, in the same manner, by joining AE , BK , it is demonstrated that the parallelogram CL is equal to the square HC . Therefore the whole square $BDEC$ is equal to the two squares GB , HC ; and the square $BDEC$ is described upon the straight line BC , and the squares GB , HC upon BA , AC : Wherefore the square upon the side BC is equal to the squares upon the sides BA , AC . Therefore, in any right angled triangle, &c. Q. E. D.

Book I.

f 4. 1.

g 41. 1.

h 6 Ax.

PROP. XLVIII. THEOR.

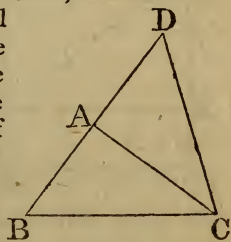
IF the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

If the square described upon BC , one of the sides of the triangle ABC , be equal to the squares upon the other sides BA , AC , the angle BAC is a right angle.

From the point A draw ^a AD at right angles to AC , and make AD equal to BA , and join DC : Then, because DA is equal to AB , the square of DA is equal to the square of AB : To each of these add the square of AC ; therefore the squares of DA , AC , are equal to the squares of BA , AC : But the square of DC is equal ^b to the squares of DA , AC , because DAC is a right angle; and the square of BC , by hypothesis, is equal to the squares of BA , AC ; therefore, the square of DC is equal to the square of BC ; and therefore

a 11. 1.

b 47. 1.



therefore

Book I.

therefore also the side DC is equal to the side BC . And because the side DA is equal to AB , and AC common to the two triangles DAC , BAC , and the base DC likewise equal to the base BC , the angle DAC is equal to the angle BAC : But DAC is a right angle; therefore also BAC is a right angle. Therefore, if the square, &c. Q. E. D.

c s. r.

E L E



E L E M E N T S

O F

G E O M E T R Y.

B O O K II.

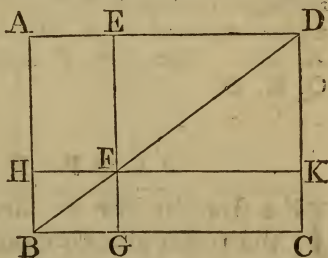
D E F I N I T I O N S.

I.

EVERY right angled parallelogram, or *rectangle*, is said to be contained by any two of the straight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. ' Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.'



E

P R O P.

PROP. I. THEOR.

IF there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the straight lines A, BC is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.

a 11. 1.

From the point B draw a

b 3. 1.

BF at right angles to BC, and

c 31. 1.

make BG equal b to A; and

through G draw c GH paral-

lel to BC; and through D,

E, C, draw c DK, EL, CH

parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for

it is contained by GB, BC, and GB is equal to A; and BK

is contained by A, BD, for it is contained by GB, BD, and

which GB is equal to A; and DL is contained by A, DE, be-

d 34. 1.

cause DK, that is, d BG, is equal to A; and in like manner

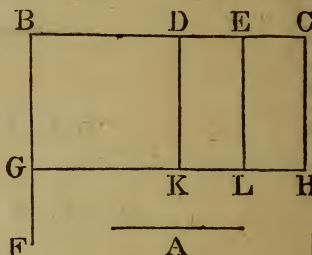
the rectangle EH is contained by A, EC: Therefore the

rectangle contained by A, BC is equal to the several rectangles

contained by A, BD, and by A, DE, and also by A, EC.

Therefore, if there be two straight lines, &c.

Q. E. D.

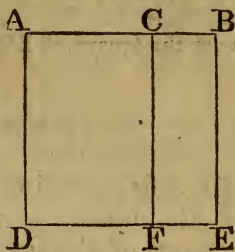


PROP. II. THEOR.

IF a straight line be divided into any two parts the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.

Let the straight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle * AB, AC, shall be equal to the square of AB.

Upon AB describe ^a the square ADEB, and through C draw ^b CF, parallel to AD or BE: then AE is equal to the rectangles AF, CE; and AE is the square of AB; and AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB: therefore the rectangle contained by AB, AC together with the rectangle AB, BC, is equal to the square of AB. Therefore a straight line, &c. Q. E. D.



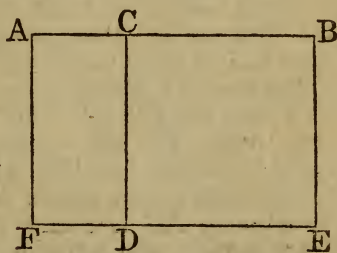
a 46. I.
b 31. I.

PROP. III. THEOR.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the foregoing part.

Let the straight line AB be divided into two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB, together with the square of BC.

Upon BC describe ^a the square CDEB, and produce AD to F, and through A draw ^b AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE. Now AE is the rectangle contained by AB, C, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is



a 46. I.
b 31. I.

E 2 equal

* N. B. To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines AB, AC is sometimes simply called the rectangle AB, AC.

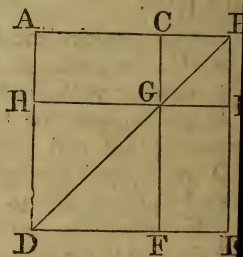
Book II equal to CB; and DB is the square of BC: therefore the rectangle AB, BC is equal to the rectangle AC, CB together with the square of BC. If therefore a straight, &c. Q. E. D.

PROP. IV. THEOR.

IF a straight line be divided into any two parts the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts. ✓

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB, and to twice the rectangle contained by AC, CB.

- a 46. I. Upon AB describe a the square ADEB, and join BD, and
 b 31. I. through C draw b CGF parallel to AD or BE, and through G draw HK parallel to AB or DE. And because CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal c to the interior and opposite angle ADB; but
 c 29. I. ADB is equal d to the angle ABD, because BA is equal to AD, being sides of a square; wherefore the angle CGB is equal to the angle GBC; and therefore the side BC is equal
 e 6. I. to the side CG: but CB is equal f also to GK, and CG to BK; wherefore the figure CGKB is equilateral. It is likewise rectangular; for the angle CBK being a right angle, the other angles of the parallelogram CGKB are also right angles g.
 f 34. I. Wherefore CGKB is a square, and it is upon the side CB. For the same reason HF also is a square, and it is upon the side HG, which is equal to AC; therefore HF, CK are the squares of AC, CB. And because the complement AG is equal h to the complement GE, and because AG is the rectangle contained by AC, CG, the rectangle AG is equal to the rectangle AC, CB; GE is also equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB. And HF, CK are the squares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the squares of AC, CB.



✕

$$(A+B)^2 = A^2 + 2AB + B^2$$

AC, CB, and to twice the rectangle AC, CB: But HF, CK, Book II.
 AG, GE make up the whole figure ADEB, which is the
 square of AB: Therefore the square of AB is equal to the
 squares of AC, CB, and twice the rectangle AC, CB.
 Wherefore if a straight line, &c. Q. E. D.

COR. From the demonstration, it is manifest that the pa-
 rallelograms about the diameter of a square are likewise
 squares.

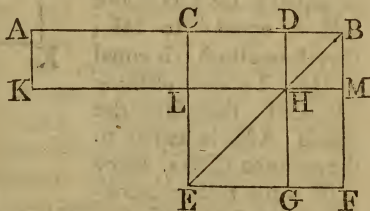
P R O P. V. T H E O R.

IF a straight line be divided into two equal parts,
 and also into two unequal parts; the rectangle
 contained by the unequal parts, together with the
 square of the line between the points of section, is
 equal to the square of half the line.

Let the straight line AB be divided into two equal parts in
 the point C, and into two unequal parts at the point D; the
 rectangle AD, DB, together with the square of CD, is equal
 to the square of CB.

Upon CB describe a the square CEFB, join BE, and a 46. I.
 through D draw b DHG parallel to CE or BF; and through b 31. I.
 H draw KLM parallel to CB or EF; and also through A
 draw AK parallel to CL or BM: And because the comple-
 ment CH is equal c to the complement HF, to each of these c 43. I.
 add DM; therefore the whole CM is equal to the whole DF;
 but CM is equal d to AL, d 36. I.

because AC is equal to CB; therefore also AL is
 equal to DF. To each of
 these add CH, and the
 whole AH is equal to DF
 and CH: but AH is the
 rectangle contained by AD,
 DB, for DH is equal e to
 DB; and DF together with CH is the gnomon CMG;
 therefore the gnomon CMG is equal to the rectangle AD,
 DB. To each of these add LG, which is equal e to the
 square of CD; therefore the gnomon CMG, together with
 LG, is equal to the rectangle AD, DB, together with the
 square



e Cor 4. 2.

Book II. square of CD : But the gnomon CMG and LG make up the whole figure $CEFB$, which is the square of CB : Therefore the rectangle AD , DB , together with the square of CD is equal to the square of CB . Wherefore, if a straight line &c. **Q. E. D.**

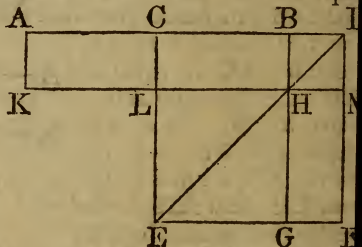
From this proposition it is manifest, that the difference of the squares of two unequal lines AC , CD , is equal to the rectangle contained by their sum and difference.

PROP. VI. THEOR.

IF a straight line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C , and produced to the point D ; the rectangle AD , DB , together with the square of CB , is equal to the square of CD .

Upon CD describe ^a the square $CEFD$, join DE , and through B draw ^b BHG parallel to CE or DF , and through H draw KLM parallel to AD or EF , and also through A draw AK parallel to CL or DM : and because AC is equal to CB , the rectangle AL is equal ^c to CH ; but CH is equal ^d to HF ; therefore also AL is equal to HF : To each of these add CM ; therefore the whole AM is equal to the gnomon CMG . Now AM is the rectangle con-



^e Cor. 4. 2. tained by AD , DB , for DM is equal ^e to DB : therefore the gnomon CMG is equal to the rectangle AD , DB ; add to each of these LG , which is equal to the square of CB : therefore the rectangle AD , DB , together with the square of CB , is equal to the gnomon CMG , together with LG : But the gnomon CMG together with LG makes up the whole figure $CEFD$.

CEFD, which is the square of CD; therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD. Wherefore, if a straight line, &c. Q. E. D. Book II.

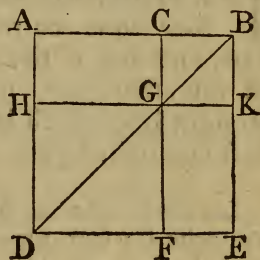
PROP. VII. THEOR.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C; the squares of AB, BC are equal to twice the rectangle AB, BC together with the square of AC.

Upon AB describe ^a the square ADEB, and construct the figure as in the preceding propositions: And because AG is equal ^b to GE, add to each of them CK; the whole AK is ^b 43. 1.

Therefore equal to the whole CE; therefore AK, CE, are double of CK: But AK, CE are the gnomon AKF together with the square CK; therefore the gnomon AKF, together with the square CK, is double of AK. But twice the rectangle AB, C is also double of AK, for BK equal ^c to BC: therefore the gnomon AKF, together with the



^c Cor. 4. 2.

square CK, is equal to twice the rectangle AB, BC. To each of these equals add HF, which is equal to the square of AC; therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC and the square of AC: now the gnomon AKF, together with the squares CK, HF, makes up the whole figure ADEB and CK, which are the squares of AB and BC: Therefore the squares of AB and BC are equal to twice the rectangle AB, C, together with the square of AC. Wherefore, if a straight line, &c. Q. E. D.

Book I.

Otherwise,

- a 4. 2. " Because the square of AB is equal to the squares of AC
 " and CB, ^a together with twice
 " the rectangle ACB, if the
 " square of CB be added to both, 
 " the squares of AB and CB are equal to the square of
 " AC together with twice the square of CB and twice the
 b 3. 2. " rectangle ACB. But the square of CB together with the
 " rectangle ACB is equal to the ^b rectangle ABC; and
 " therefore twice the square of CB, together with twice the
 " rectangle ACB, is equal to twice the rectangle ABC. Where-
 " fore the squares of AB and CB are equal to twice the rect-
 " angle ACB, together with the square of AC. Q. E. D.
 " Cor. Hence, the sum of the squares of any two lines is
 " equal to twice the rectangle contained by the lines together
 " with the square of the difference of the lines."

P R O P. VIII. T H E O R.

IF a straight line be divided into any two parts,
 four times the rectangle contained by the whole
 line, and one of the parts, together with the square
 of the other part, is equal to the square of the
 straight line which is made up of the whole and the
 first mentioned part.

Let the straight line AB be divided into any two parts in
 the point C; four times the rectangle AB, BC, together
 with the square of AC, is equal to the square of the straight
 line made up of AB and BC together.

- Produce AB to D, so that BD be equal to CB, and upon
 AD describe the square AEFD; and construct two figures
 such as in the preceding. Because CB is equal to BD, and
 that CB is equal ^a to GK, and BD to KN; therefore GK is
 a 34. I. equal to KN: For the same reason, PR is equal to RO; and
 because CB is equal to BD, and GK to KN, the rectangle
 CK is equal ^b to BN, and GR to RN: But CK is equal ^c to
 b 36. I. RN, because they are the complements of the parallelogram
 c 43. I. CO; therefore also BN is equal to GR; and the four rect-
 angles BN, CK, GR, RN are therefore equal to one another,

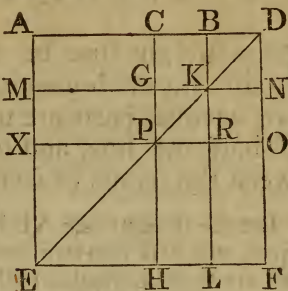
and

and so are quadruple of one of them, CK. Again, because Book II.
 CB is equal to BD, and BD equal^d to BK, that is, to CG; d Cor. 4. 2.
 and CB equal to GK, that^d is, to GP; therefore CG is equal to GP; and because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF: But MP is equal^e to PL, because they are the complements of the parallelogram ML; e 43. I.
 wherefore AG is equal also to RF: Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so are quadruple of one of them, AG. And it was demonstrated, that the four CK, BN, GR, and RN are quadruple of CK: therefore the eight rectangles which make up the gnomon AOH, are quadruple of AK. And because AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC, is quadruple of AK: But the gnomon AOH was demonstrated to be quadruple of AK; therefore four times the rectangle AB, BC, is equal to the gnomon AOH. To each of these add XH, which is equal^d to the square of AC; d Cor. 4. 2.
 therefore four times the rectangle AB, BC together with the square of AC, is equal to the gnomon AOH and the square XH: But the gnomon AOH and the square XH make up the figure AEFD, which is the square of AD: Therefore four times the rectangle AB, BC, together with the square of AC, is equal to the square of AD, that is, of AB and BC added together in one straight line. Wherefore, if a straight line, &c. Q. E. D.

“COR. 1. Hence, because AD is the sum, and AC the difference of the lines AB and BC, four times the rectangle contained by any two lines together with the square of their difference, is equal to the square of the sum of the lines.”

“COR. 2. From the demonstration it is manifest, that since the square of CD is quadruple of the square of CB, the square of any line is quadruple of the square of half that line.”

P R O P.

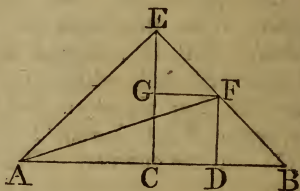


PROP. IX. THEOR.

IF a straight line be divided into two equal, and also into two unequal parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let the straight line AB be divided at the point C into two equal, and at D into two unequal parts: The squares of AD, DB are together double of the squares of AC, CD.

- a 11. 1. From the point C draw ^a CE at right angles to AB, and make it equal to AC or CB, and join EA, EB; through D
- b 31. 1. draw ^b DF parallel to CE, and through F draw FG parallel to AB; and join AF: Then, because AC is equal to CE, the
- c. 5. 1. angle EAC is equal ^c to the angle AEC; and because the angle ACE is a right angle, the two others AEC, EAC together make one right angle ^d; and they are equal to one another; each of them therefore
- d. 32. 1. is half of a right angle. For the same reason each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle: And because the angle GEF is half a right angle, and EGF a
- e. 29. 1. right angle, for it is equal ^e to the interior and opposite angle ECB, the remaining angle EFG is half a right angle; therefore the angle GEF is equal to the angle EFG, and the side
- f 6. 1. EG equal ^f to the side GF: Again, because the angle at B is half a right angle and FDB a right angle, for it is equal ^e to the interior and opposite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the side DF to ^f the side DB. Now because AC is equal to CE, the square of AC is equal to the square of CE; therefore the squares of AC, CE, are double of the square of AC: But the square of EA is equal ^g to the squares of AC, CE, because ACE is a right angle; therefore the square of EA is double of the square of AC: Again, because EG is equal to GF, the square of EG is equal to the square of GF; therefore the squares of EG, GF are double of the
- g 47. 1. the



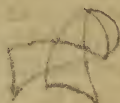
the square of GF ; but the square of EF is equal to the squares of EG , GF ; therefore the square of EF is double of the square GF ; and GF is equal ^h to CD ; therefore the square of EF is double of the square of CD : But the square of AE is likewise double of the square of AC ; therefore the squares of AE , EF are double of the squares of AC , CD : And the square of AF is equal ⁱ to the squares of AE , EF , because AEF is a right angle; therefore the square of AF is double of the squares of AC , CD : But the squares of AD , DF are equal to the square of AF , because the angle ADF is a right angle; therefore the squares of AD , DF are double of the squares of AC , CD : And DF is equal to DB ; therefore the squares of AD , DB are double of the squares of AC , CD . If therefore a straight line, &c. Q. E. D.

Book II.

h 34. 1.

i 47. 1.

PROP. X. THEOR.



IF a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C , and produced to the point D ; the squares of AD , DB are double of the squares of AC , CD .

From the point C draw ^a CE at right angles to AB : And make it equal to AC or CB , and join AE , EB ; through E draw ^b EF parallel to AB , and through D draw DF parallel to CE : And because the straight line EF meets the parallels EC , FD , the angles CEF , EFD are equal ^c to two right angles; and therefore the angles BEF , EFD are less than two right angles: But straight lines, which with another straight line make the interior angles upon the same side less than two right angles, do meet ^d if produced far enough: Therefore EB , FD shall meet, if produced towards B , D ; let them meet in G , and join AG : Then, because AC is equal to CE , the angle CEA is equal ^e to the angle EAC ; and the angle ACE is a right angle; therefore each of the angles CEA , EAC is half

a 11. 1.

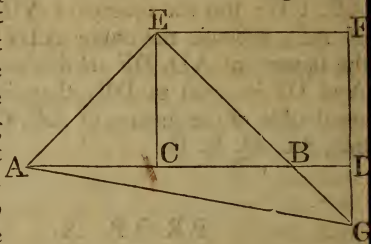
b 31. 1.

c 29. 1.

d Cor. 29. 1.

e 5. 1.

Book II. half a right angle f : For the same reason, each of the angles
 f 32. I. CEB, EBC is half a right angle; therefore AEB is a right
 angle: And because EBC is half a right angle, DBG is also
 g 15. I. half a right angle, for they are vertically opposite; but
 c 29. I. BDG is a right angle, because it is equal c to the alternate
 angle DCE; therefore the remaining angle DGB is half a
 right angle, and is therefore equal to the angle DBG; where-
 h 6. I. fore also the side DB is equal h to the side DG: Again, be-
 cause EGF is half a right
 angle, and that the angle at
 i 34. I. F is a right angle, because
 it is equal i to the opposite
 angle ECD, the remaining
 angle FEG is half a right
 angle, and equal to the
 angle EGF; wherefore also
 the side GF is equal h to the
 side FE. And because EC is equal to CA, the square of EC
 is equal to the square of CA; therefore the squares of EC,
 CA are double of the square of CA: But the square of EA
 k 47. I. is equal k to the squares of EC, CA; therefore the square of
 EA is double of the square of AC: Again, because GF is
 equal to FE, the square of GF is equal to the square of FE,
 and therefore the squares of GF, FE are double of the square
 of EF: But the square of EG is equal k to the squares of GF,
 FE; therefore the square of EG is double of the square of
 EF; and EF is equal to CD; wherefore the square of EG is
 double of the square of CD; but it was demonstrated, that
 the square of EA is double of the square of AC; therefore
 the squares of AE, EG are double of the squares of AC, CD:
 And the square of AG is equal k to the squares of AE, EG;
 therefore the square of AG is double of the squares of AC,
 CD: But the squares of AD, DG are equal k to the square of
 AG; therefore the squares of AD, DG are double of the
 squares of AC, CD: But DG is equal to DB; therefore the
 squares of AD, DB are double of the squares of AC, CD:
 Wherefore, if a straight line, &c. Q. E. D.



PROP. XI. PROB.

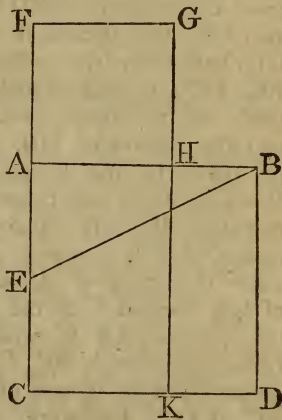
TO divide a given straight line into two parts, so that the rectangle contained by the whole, and one of the parts, shall be equal to the square of the other part.

Let AB be the given straight line; it is required to divide it into two parts, so that the rectangle contained by the whole, and one of the parts, shall be equal to the square of the other part.

Upon AB describe ^a the square ABDC; bisect ^b AC in E, and join BE; produce CA to F, and make ^c EF equal to EB, and upon AF describe ^a the square FGHA; AB is divided in H, so that the rectangle AB, BH is equal to the square of AH.

Produce GH to K: Because the straight line AC is bisected in E, and produced to the point F, the rectangle CF, FA, together with the square of AE, is equal ^d to the square of EF: But EF is equal to EB; therefore the rectangle CF, FA, together with the square of AE, is equal to the square of EB:

And the squares of BA, AE are equal ^e to the square of EB, because the angle EAB is a right angle; therefore the rectangle CF, FA, together with the square of AE, is equal to the squares of BA, AE: take away the square of AE, which is common to both, therefore the remaining rectangle CF, FA is equal to the square of AB. Now the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the square of AB; therefore FK is equal to AD: take away the common part AK, and the remainder FH is equal to the remainder HD: And HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the square of AH; therefore the rectangle AB, BH



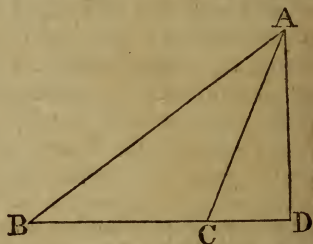
Book II. is equal to the square of AH: Wherefore the straight line AB is divided in H so, that the rectangle AB, BH is equal to the square of AH. Which was to be done.

PROP. XII. THEOR.

IN obtuse angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

a 12. 1. Let ABC be an obtuse angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn a perpendicular to BC produced: The square of AB is greater than the squares of AC, CB, by twice the rectangle BC, CD.

b 4. 2. Because the straight line BD is divided into two parts in the point C, the square of BD is equal^b to the squares of BC, CD, and twice the rectangle BC, CD: to each of these equals add the square of DA; and the squares of DB, DA are equal to the squares of BC, CD, DA, and twice the rectangle BC, CD: But the square of BA is equal^c to the squares of BD, DA, because the angle at D is a right angle; and the square of CA is equal^c to the squares of CD, DA: Therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC, CD; that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuse angled triangles, &c. Q. E. D.



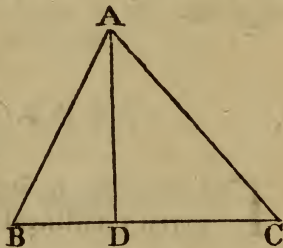
PROP.

PROP. XIII. THEOR.

IN every triangle, the square of the side subtending any of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle; and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC , one of the sides containing it, let fall the perpendicular AD from the opposite angle: The square of AC , opposite to the angle B , is less than the squares of CB , BA by twice the rectangle CB , BD . a 12. 1.

First, Let AD fall within the triangle ABC ; and because the straight line CB is divided into two parts in the point D , the squares of CB , BD are equal b to twice the rectangle contained by CB , BD , and the square of DC : To each of these equals add the square of AD ; therefore the squares of CB , BD , DA are equal to twice the rectangle CB , BD , and the squares of AD , DC : But the square of AB is equal c to the squares of BD , DA , because the angle BDA is a right angle; and the square of AC is equal to the squares of AD , DC : Therefore the squares of CB , BA are equal to the square of AC , and twice the rectangle CB , BD , that is, the square of AC alone is less than the squares of CB , BA by twice the rectangle CB , BD . b 7. 2.



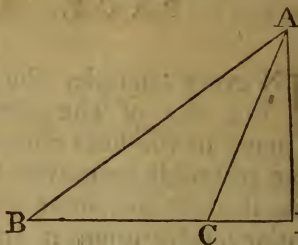
Secondly, Let AD fall without the triangle ABC : Then, because the angle at D is a right angle, the angle ACB is greater d than a right angle; and therefore the square of AB is equal e to the squares of AC , CB , and twice the rectangle BC , CD : To these equals add the square of BC , and the squares of AB , BC are equal to the square of AC , and twice the square of BC , and twice the rectangle BC , CD : c 47. 1.

CD : d 16. 1.
e 12. 2.

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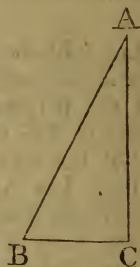
f 3. 2.

CD: But because BD is divided into two parts in C, the rectangle DB, BC is equal ^f to the rectangle BC, CD and the square of BC: And the doubles of these are equal: Therefore the squares of AB, BC are equal to the square of AC, and twice the rectangle DB, BC: Therefore the square of AC alone is less than the squares of AB, BC by twice the rectangle DB, BC.



g 47. 1.

Lastly, Let the side AC be perpendicular to BC; then is BC the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares of AB, BC are equal ^g to the square of AC, and twice the square of BC: Therefore, in every triangle, &c. Q. E. D.



PROP. XIV. PROB.

TO describe a square that shall be equal to given rectilineal figure.

Let A be the given rectilineal figure; it is required to describe a square that shall be equal to A.

a 45. 1.

Describe ^a the rectangular parallelogram BCDE equal to the rectilineal figure A. If then the sides of it BE, ED are equal to one another, it is a square, and what was required is done; but if they are not equal, produce one of them BE to F, and make EF equal to ED, and bisect BF in G: and from the centre G, at the distance GB, or GF, describe the semicircle BHF, and produce DE to H, and join GH. Therefore, because the straight line BF is divided into two equal parts in the point G, and into two unequal in the point H, the rectangle BE, EF, together with the square of EG,

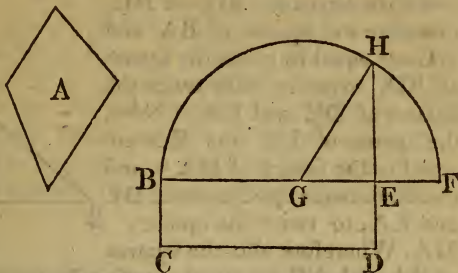
b 5. 2.

equal ^b to the square of GF: but GF is equal to GH; therefore

for

fore the rectangle BE, EF, together with the square of EG, is equal to the square of GH: But the squares of HE, EG are equal to the square of GH: Therefore, the rectangle BE, EF together with the square of EG, is equal to the squares of HE, EG. Take away the square of EG, which is

Book II.
c 47.1.



common to both, and the remaining rectangle BE, EF is equal to the square of EH: But the rectangle contained by BE, EF is the parallelogram BD, because EF is equal to ED; therefore BD is equal to the square of EH; but BD is equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the square of EH: Wherefore a square has been made equal to the given rectilineal figure A, viz. the square described upon EH. Which was to be done.

PROP. A. THEOR.

IF one side of a triangle be bisected, the sum of the squares of the other two sides is double of the square of half the side bisected, and of the square of the line drawn from the point of bisection to the opposite angle of the triangle. See N.

Let ABC be a triangle, of which the side BC is bisected in D, and DA drawn to the opposite angle; the squares of BA and AC are together double of the squares of BD and DA.

From A draw AE perpendicular to BC, and because EA is a right angle, the square of AB is equal to the squares of BE^a and EA; for the same reason, the square of AC is equal to the two squares of CE and EA. Therefore, the squares of BA and AC are equal to the squares of BE and EC, together with twice the square of EA. But because the

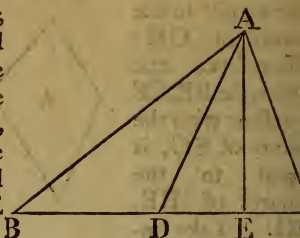
a 47.1.

F

line

Book II.
b 9. 2.

line BC is cut equally in D , and unequally in E , the square of BE and EC ^b are equal to twice the squares of BD and DE ; therefore the squares of BA and AC are equal to twice the square of BD , together with twice the squares of DE and EA . Now, the squares of DE and EA are equal to the square of DA ^a, and therefore twice the squares of DE and EA , to twice the square of DA . Wherefore also, the squares of BA and AC are equal to twice the square of BD , together with twice the square of DA . Therefore, &c. Q. E. D.



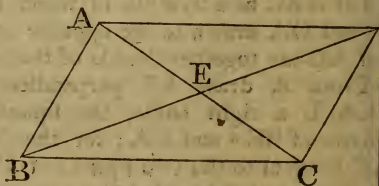
PROP. B. THEOR.

THE sum of the squares of the diameters of any parallelogram is equal to the sum of the squares of the sides of the parallelogram.

Let $ABCD$ be a parallelogram, of which the diameters AC and BD ; the sum of the squares of AC and BD is equal to the sum of the squares of AB , BC , CD , DA .

Let AC and BD intersect one another in E : and because the vertical angles AED , CEB are equal ^a, and also the alternate angles EAD , ECB ^b, the triangles ADE , CEB have two angles in the one equal to two angles in the other, each to each: but the sides AD and BC , which are opposite equal angles in these triangles, are also equal ^c; therefore the other sides which are opposite to the equal angles are equal, viz. AE to EC , and ED to EB .

Since, therefore, BD is bisected in E , the squares of BA and AD are equal to twice the square of BE , together with twice the square of EA ^e; and for the same reason the square of BC and CD



equal to twice the square of BE, together with twice the square of EC, that is, of EA, because EC is equal to EA. Therefore the four squares of BA, AD, DC, CB are equal four times the squares of BE and of EA. But the square BD is equal to four times the square of BE, because BD is double of BE^e; and for the same reason, the square of AC is equal to four times the square of AE: Wherefore also, the squares of BD and AC are equal to the four squares of BA, AD, DC, CB. Therefore the sum of the squares, &c. E. D.

Book II.

e Cor. 8. 2.

COR. From the demonstration, it is manifest that the diagonals of every parallelogram bisect one another.

F 2

E L E.

ELEMENTS

OF

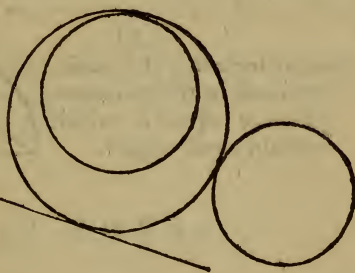
GEOMETRY.

BOOK III.

DEFINITIONS.

I.

A straight line is said to touch a circle, when it meets the circle, and being produced does not cut it.



Book III.

II.

Circles are said to touch one another, which meet, but do not cut one another.

III.

Right lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



IV.

The straight line on which the greater perpendicular falls, is said to be farther from the centre.

F 3

V.

Book III.

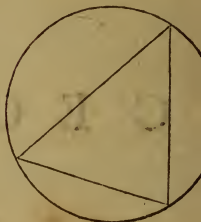
V.

A segment of a circle is the figure contained by a straight line and the circumference it cuts off.



VI.

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.

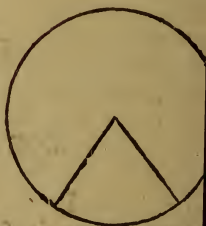


VII.

And an angle is said to infist or stand upon the circumference intercepted between the straight lines that contain the angle.

VIII.

The sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



XI.

Similar segments of a circle, are those in which the angles are equal, or which contain equal angles.



PROP. I. PROB.

TO find the centre of a given circle.

Let AEC be the given circle; it is required to find centre.

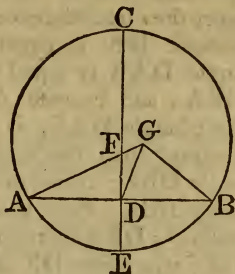
D

Draw within it any straight line AB, and bisect ^a it in D; From the point D draw ^b DC at right angles to AB, and produce it to E, and bisect CE in F: The point F is the centre of the circle ABC.

Book III.

^a 10. I.^b 11. I.

For, if it be not, let, if possible, G be the centre, and join GA, GD, GB: Then, because DA is equal to DB, and DG common to the two triangles ADG, BDG, the two sides AD, DG are equal to the two BD, DG, each to each; and the base GA is equal to the base GB, because they are drawn from the centre G*: therefore the angle ADG is equal ^c to the angle GDB: But when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle ^d.

^c 8. I.^d 7. def. I.

Therefore the angle GDB is a right angle: But FDB is likewise a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the less, which is impossible: Therefore G is not the centre of the circle ABC: In the same manner it can be shown, that no other point but F is the centre; that is, F is the centre of the circle ABC: Which was to be found.

COR. From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

F 4

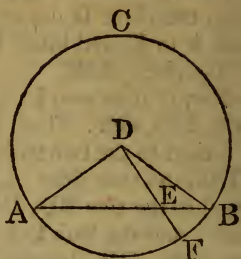
Let

N. B. Whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that they are drawn from the circumference.

Book III.

Let ABC be a circle, and A, B any two points in the circumference; the straight line drawn from A to B shall fall within the circle.

Take any point in AB as E ; find D the centre of the circle ABC , join AD, DB and DE , and let DE , meet the circumference in F . Then because DA is equal to DB , the angle DAB is equal^a to the angle DBA ; and because AE , a side of the triangle DAE , is produced to B , the angle DEB is greater^b than the angle DAE ; but DAE is equal to the angle DBE ; therefore the angle DEB is greater than the angle DBE : Now to the greater angle the greater side is opposite^d; DB is therefore greater than DE : but DB is equal to DF ; wherefore DF is greater than DE , and the point E is therefore within the circle. The same may be demonstrated of any other point between A and B , therefore AB is within the circle. Wherefore, if any two points, &c. Q. E. D.



PROP. III. THEOR.

IF a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and, if it cuts it at right angles, it shall bisect it.

Let ABC be a circle, and let CD , a straight line drawn through the centre, bisect any straight line AB , which does not pass through the centre, in the point F : It cuts it also at right angles.

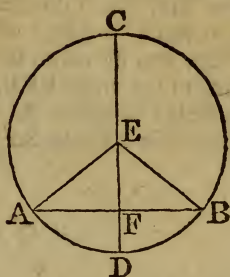
2 1. 3.

Take^a E the centre of the circle, and join EA, EB . Then, because AF is equal to FB , and FE common to the two triangles

Book III.

b 8. r.

c 7. def. 1.



Triangles AFE, BFE, there are two sides in the one equal to two sides in the other, and the base EA is equal to the base EB; therefore the angle AFE is equal to the angle BFE: But when a straight line, standing upon another makes the adjacent angles equal to one another, each of them is a right angle: Therefore each of the angles AFE, BFE is a right angle; wherefore the straight line CD, drawn through the centre bisecting another AB that does not pass through the centre, cuts the same at right angles.

But let CD cut AB at right angles; CD also bisects it, that is, AF is equal to FB.

The same construction being made, because EA, EB from the centre are equal to one another, the angle EAF is equal to the angle EBF; and the right angle AFE is equal to the right angle BFE: Therefore, in the two triangles EAF, EBF, there are two angles in one equal to two angles in the other, and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal; AF therefore is equal to FB. Wherefore, if a straight line &c. Q. E. D.

d 5. r.

e 26. r.

PROP. IV. THEOR.

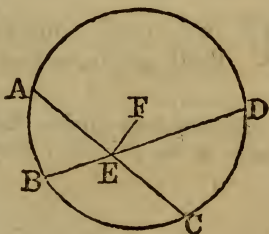
IF in a circle two straight lines cut one another which do not both pass through the centre, they do not bisect each the other.

Let ABCD be a circle, and AC, BD two straight lines in it, which cut one another in the point E, and do not both pass through the centre: AC, BD do not bisect one another.

For, if it is possible let AE be equal to EC, and BE to ED: If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre. But if neither of them pass through the centre, take a F the centre of the circle, and join EF: and because FE, a straight line through the centre, bisects another AC which does not pass through the centre, it shall cut it at right angles; wherefore FEA is a right

a 1. 3.

b 3. 3.



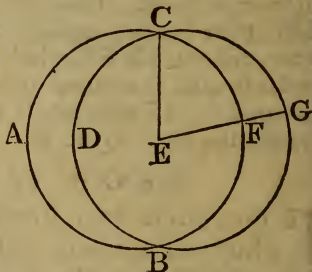
Book III. angle : Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, it shall cut it at right ^b angles ; wherefore FEB is a right angle : and FEA was shown to be a right angle ; therefore FEA is equal to the angle FEB, the less to the greater, which is impossible : therefore AC, BD do not bisect one another. Wherefore, if in a circle, &c. Q. E. D.

PROP. V. THEOR.

IF two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C ; they have not the same centre.

For, if it be possible, let E be their centre : join EC, and draw any straight line EFG meeting them in F and G : and because E is the centre of the circle ABC, CE is equal to EF : again, because E is the centre of the circle CDG CE is equal to EG : but, CE was shown to be equal to EF, therefore EF is equal to EG, the less to the greater, which is impossible : therefore E is not the centre of the circles ABC, CDG. Wherefore, if two circles, &c. Q. E. D.

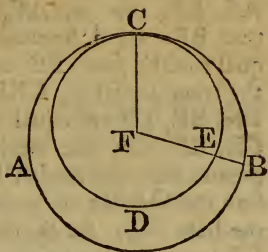


PROP. VI. THEOR.

IF two circles touch one another internally, they shall not have the same centre.

Let the two circles ABC, CDE, touch one another internally in the point C : they have not the same centre.

For, if they can, let it be F ; join FC , and draw any straight line FEB meeting them in E and B ; and because F is the centre of the circle ABC , CF is equal to FB ; also, because F is the centre of the circle CDE , CF is equal to FE : and CF was shown equal to FB ; therefore FE is equal to FB , the less to the greater, which is impossible; wherefore F is not the centre of the circles ABC , CDE . Therefore, if two circles, &c. Q. E. D.



PROP. VII. THEOR.

IF any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: And from the same point there can be drawn only two straight lines that are equal to one another, one upon each side of the shortest line.

Let $ABCD$ be a circle, and AD its diameter, in which let any point F be taken which is not the centre: let the centre be E ; of all the straight lines FB , FC , FG , &c. that can be drawn from F to the circumference, FA is the greatest, and FD , the other part of the diameter AD , is the least: and of the others, FB is greater than FC , and FC than FG .

Join BE , CE , GE ; and because two sides of a triangle are greater than the third, BE , EF are greater than BF ; a 20. 1.
but

Book III. but AE is equal to EB; therefore

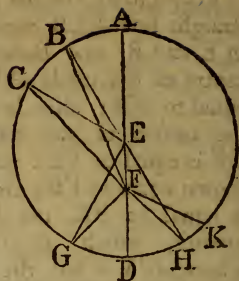
AE, EF, that is, AF, is greater than BF: again, because BE is equal to CE, and FE common to the triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF; but the angle BEF is greater than the angle CEF; therefore the base BF is greater^b than the base FC: for the same reason, CF is greater than GF: again, because GF, FE are greater^a than EG, and EG is equal to ED; GF, FE are greater than ED: take away the common part FE, and the remainder GF is greater than the remainder FD: therefore FA is the greatest, and FD the least of all the straight lines from F to the circumference; and BF is greater than CF, and CF than GF.

b 24. I.

Also there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the shortest line FD: at the point E in the straight line EF, make^c the angle FEH equal to the angle GEF, and join FH: Then, because GE is equal to EH, and EF common to the two triangles GEF, HEF; the two sides GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; therefore the base FG is equal^d to the base FH: but besides FH, no straight line can be drawn from F to the circumference equal to FG: for, if there can, let it be FK; and because FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which passes through the centre, is equal to one which is more remote, which is impossible. Therefore, if any point be taken, &c. Q. E. D.

c 23. I.

d 4. I.



PROP. VII. THEOR.

IF any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to that through the centre is always greater than the more remote: But of those which fall upon the convex circumference, the least is that between the point without the circle, and the diameter; and of the rest, that which is nearer to the least is always less than the more remote: And only two equal straight lines can be drawn from the point unto the circumference, one upon each side of the least.

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DG be drawn to the circumference, whereof DA passes through the centre. Of those which fall upon the concave part of the circumference AEFC, the greatest is AD which passes through the centre; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DG: but of those which fall upon the convex circumference HLKG, the least is DG between the point D and the diameter AG; and the nearer to it is always less than the more remote, viz. DK than DL, and DL than DH.

Take ^a M the centre of the circle ABC, and join ME, MF, ^{a 1. 3.} MC, MK, ML, MH: And because AM is equal to ME, add MD to each, therefore AD is equal to EM, MD; but EM, MD are greater ^b than ED; therefore also AD is greater ^{b 20. 1.} than ED. Again, because ME is equal to MF, and MD common

Book III. common to the triangles EMD, FMD; EM, MD are equal to FM, MD; but the angle EMD is greater than the angle FMD; therefore the base ED

c 24. I. is greater ^c than the base FD:

In like manner it may be shewn that FD is greater than CD:

Therefore DA is the greatest; and DE greater than DF, and

DF than DC. And because MK, KD are greater ^b than MD, and MK is equal to MG,

d 4. Ax. the remainder KD is greater ^d than the remainder GD, that is

GD is less than KD: And because MK, DK are drawn to the

point K within the triangle MLD from M, D, the extremities of its side MD; MK, KD

c 21. I. are less ^c than ML, LD, whereof MK is equal to ML; therefore the remainder DK is less than the remainder DL:

In like manner, it may be shewn that DL is less than DH:

Therefore DG is the least, and DK less than DL, and DL than DH.

Also there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the least: at the point M, in the straight line MD,

make the angle DMB equal to the angle DMK, and join DB; and because in the triangles KMD, BMD, the side KM is

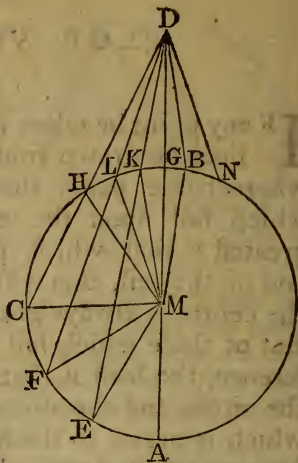
equal to the side BM, and MD common to both, and also the angle KMD equal to the angle BMD, the base DK is equal ^f

f 4. I. to the base DB. But, besides DB, no straight line can be drawn from D to the circumference, equal to DK: for, if

there can, let it be DN; then, because DN is equal to DK, and DK equal to DB, DB is equal to DN; that is, the nearer

to DG, the least, equal to the more remote, which has been shewn to be impossible. If, therefore, any point, &c. Q.

E. D.



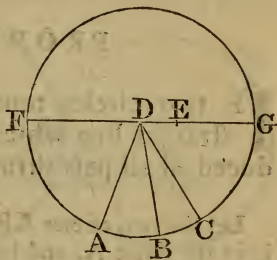
PROP.

PROP. IX. PROB.

IF a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC , from which to the circumference there fall more than two equal straight lines, viz. DA , DB , DC , the point D is the centre of the circle.

For, if not, let E be the centre, join DE and produce it to the circumference in F , G ; then FG is a diameter of the circle ABC : And because in FG , the diameter of the circle ABC , there is taken the point D which is not the centre, DG shall be the greatest line from it to the circumference, and DC greater than DB , and DB than DA ; but they are likewise equal, which is impossible: Therefore E is not the centre of the circle ABC : In like manner, it may be demonstrated, that no other point but D is the centre; D therefore is the centre. Wherefore, if a point be taken, &c. Q. E. D.



a 7. 3.

PROP. X. THEOR.

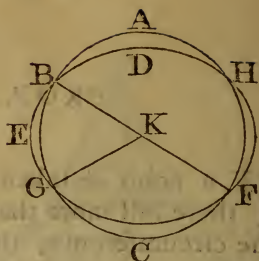
ONE circumference of a circle cannot cut another in more than two points.

If

Book III.

If it be possible, let the circumference FAB cut the circumference DEF in more than two points, viz. in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF: and because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB,

- a 9. 3. KG, KF, the point K is ^a the centre of the circle DEF; but K is also the centre of the circle ABC; therefore the same point is the centre of two circles that cut one another, which is impossible ^b. Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.

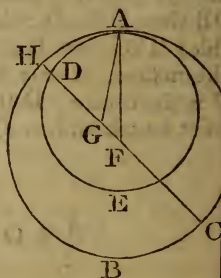


PROP. XI. THEOR.

IF two circles touch each other internally, the straight line which joins their centres being produced, shall pass through the point of contact.

Let the two circles ABC, ADE, touch each other internally in the point A, and let F be the centre of the circle ABC and G the centre of the circle ADE; the straight line which joins the centres F, G, being produced, passes through the point A.

For, if not, let it fall otherwise, if possible, as FGDH, and join AF, AG: and because AG, GF are greater ^a than FA, that is, than FH, for FA is equal to FH, both being from the same centre; take away the common part FG, and the remainder AG will be greater than the remainder GH. But AG is equal to GD, therefore GD is greater than GH; and it is also less, which is impossible. Therefore the straight line which joins the points F and G cannot fall otherwise than the point A; that is, it must pass through A. Therefore two circles, &c. Q. E. D.



PRO

PROP. XII. THEOR.

IF two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE: The straight line which joins the points F, G shall pass through the point of contact A.

For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG: and because F is the centre of the circle ABC,

FA is equal to FC:

Also because G is

the centre of the

circle ADE, AG

is equal to GD:

Therefore FA, AG

are equal to FC,

GD; wherefore the

whole FG is great-

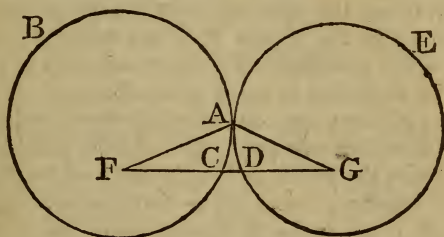
er than FA, AG; but it is also less^a, which is impossible: a 20. 1.

Therefore the straight line which joins the points F, G shall

not pass otherwise than through the point of contact A;

that is, it must pass through it. Therefore, if two circles, &c.

Q. E. D.

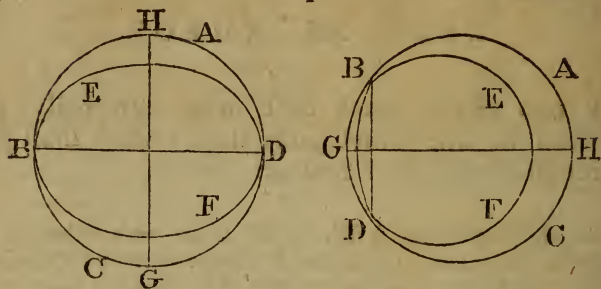


PROP. XIII. THEOR.

ONE circle cannot touch another in more points than one, whether it touches it on the inside or outside.

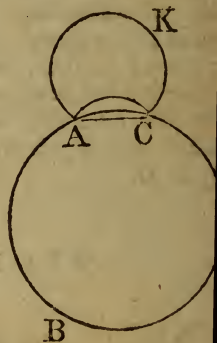
For, if it be possible, let the circle EBF touch the circle ABC in more points than one, and first on the inside, in the points B, D; join BD, and draw a GH bisecting BD at right angles: a 10. 11. 1.

Book III. angles: Therefore, because the points B, D are in the circum



ference of each of the circles, the straight line BD falls within each^b of them; and their centres are^c in the straight line GH which bisects BD at right angles: Therefore GH passes through the point of contact^d; but it does not pass through it because the points B, D are without the straight line GH which is absurd: Therefore one circle cannot touch another on the inside in more points than one.

Nor can two circles touch one another on the outside more than one point: For, if it be possible, let the circle ACK touch the circle ABC in the points A, C, and join AC. Therefore, because the two points A, C are in the circumference of the circle ACK, the straight line AC which joins them shall fall within^b the circle ACK: And the circle ACK is without the circle ABC; and therefore the straight line AC is without this last circle; but, because the points A, C are in the circumference of the circle ABC, the straight line AC must be within^b the same circle, which is absurd: Therefore one circle cannot touch another on the outside in more than one point; and it has been shewn that they cannot touch on the inside in more points than one. Therefore, one circle, &c. Q. E. D.



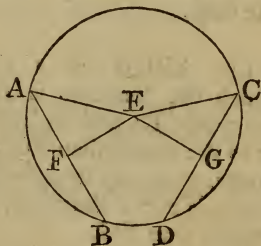
PROP. XIV. THEOR.

EQUAL straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre, are equal to one another.

Let the straight lines AB , CD , in the circle $ABDC$, be equal to one another; they are equally distant from the centre.

Take E the centre of the circle $ABDC$, and from it draw EF , EG perpendiculars to AB , CD : Then, because the straight line EF , passing through the centre, cuts the straight line AB , which does not pass through the centre, at right angles, it also bisects^a it: Wherefore AF is equal to FB , and AB double of AF . For the same reason, CD is double of CG :

And AB is equal to CD ; therefore AF is equal to CG : And because AE is equal to EC , the square of AE is equal to the square of EC : But the squares of AF , FE are equal^b to the square



a 3. 3.

of AE , because the angle AFE is a right angle; and, for the like reason, the squares of EG , GC are equal to the square of EC : Therefore the squares of AF , FE are equal to the squares of CG , GE , of which the square of AF is equal to the square of CG , because AF is equal to CG ; therefore the remaining square of FE is equal to the remaining square of EG , and the straight line EF is therefore equal to EG : But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal^c: Therefore AB , CD are equally distant from the centre.

b 47. 1.

Next, if the straight lines AB , CD be equally distant from the centre, that is, if FE be equal to EG , AB is equal to CD : For, the same construction being made, it may, as before, be

G 2

demonstrated,

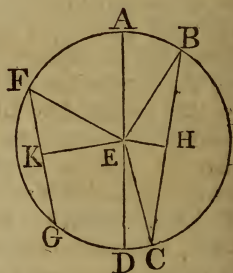
c 3. Def. 3.

Book III. demonstrated, that AB is double of AF , and CD double of CG , and that the squares of EF , FA are equal to the square of EG , GC ; of which the square of FE is equal to the square of EG , because FE is equal to EG ; therefore the remaining square of AF is equal to the remaining square of CG ; and the straight line AF is therefore equal to CG : And AB is double of AF , and CD double of CG ; wherefore AB is equal to CD . Therefore equal straight lines, &c. Q. E. D.

PROP. XV. THEOR.

THE diameter is the greatest straight line in a circle; and, of all others, that which is nearest to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let $ABCD$ be a circle, of which the diameter is AD , and the centre E ; and let BC be nearer to the centre than FG ; AD is greater than any straight line BC which is not a diameter, and BC greater than FG .



From the centre draw EH , EK perpendiculars to BC , FG , and join EB , EC , EF ; and because AE is equal to EB , and ED to EC , AD is equal to $EB + EC$: But EB , EC , are greater^a than BC ; wherefore, AD is greater than BC .

And, because BC is nearer to the centre than FG , EH is less^b than EK : But, as was demonstrated in the preceding Prop. 13, BC is double of BH , and FG double of FK , and the squares EH , HB are equal to the squares of EK , KF , of which the square of EH is less than the square of EK , because EH is less than EK ; therefore the square of BH is greater than the square of FK , and the straight line BH greater than FK ; and therefore BC is greater than FG .

Next, Let BC be greater than FG ; BC is nearer to the centre than FG ; that is, the same construction being made

EH is less than EK: Because BC is greater than FG, BH likewise is greater than KF; and the squares of BH, HE are equal to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the straight line EH less than EK. Wherefore the diameter, &c. Q. E. D.

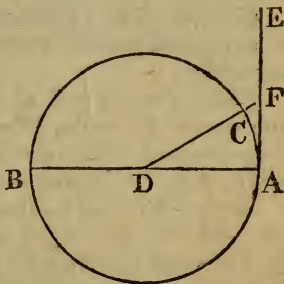
PROP. XVI. THEOR.

THE straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn between that straight line and the circumference from the extremity, so as not to cut the circle.

Let ABC be a circle, the centre of which is D, and the diameter AB: and let AE be drawn from A perpendicular to AB, AE shall fall without the circle.

In AE take any point F, join DF, and let DF meet the circle in C. Because DAF is a right angle, it is greater than the angle AFD^a; but the greater angle of any triangle is subtended by the greater side^b, therefore DF is greater than DA; and DA is equal to DC, therefore DF is greater than DC, and the point F is therefore without the circle. Now F is any point whatever on the line AE, therefore AE is without the circle.

Again, between the straight line AE and the circumference no straight line can be drawn from the point A, which does not cut the circle. Let AG be drawn, making the angle BAG less than a right angle; from D draw DH at right angles to AG; and because the angle DHA is a right angle,

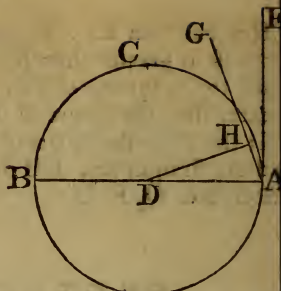


a. 32. I.

b 19. I.

Book III.

angle, each of the other angles of the triangle DAH is less than a right angle ^a; the angle DAH is therefore less than the angle DHA, and therefore also the side DH is less than the side DA ^b. The point H, therefore, is within the circle, and therefore the straight line AG cuts the circle.



COR. From this it is manifest that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it ^c. 'Also it is evident that there can be but one straight line which touches the circle in the same point.'

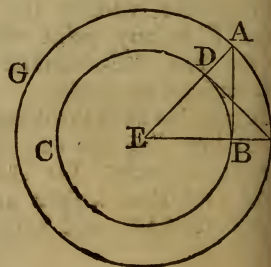
PROP. XVII. PROB.

TO draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, let A be a given point without the given circle BCD; it is required to draw a straight line from A which shall touch the circle.

^a 1. 3. Find ^a the centre E of the circle, and join AE; and from the centre E, at the distance EA, describe the circle AFG from the point D draw ^b DF at right angles to EA, and join EBF, AB. AB touches the circle BCD.

Because E is the centre of the circles BCD, AFG, EA is equal to EF: and ED to EB; therefore the two sides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the base DF is equal to the base AB, and the triangle FED to the triangle AEB, and the other angles to the



oth

other angles c : Therefore the angle EBA is equal to the Book III.
c. 4. 1.
angle EDF ; but EDF is a right angle, wherefore EBA is a
right angle; and EB is drawn from the centre: but a straight
line drawn from the extremity of a diameter, at right angles
to it, touches the circle^d: Therefore AB touches the circle; d Cor. 16. 3.
and it is drawn from the given point A . Which was to be
done.

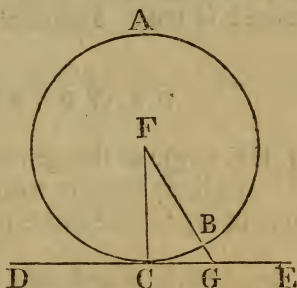
But, if the given point be in the circumference of the circle,
as the point D , draw DE to the centre E and DF at right
angles to DE ; DF touches the circle d .

PROP. XVIII. THEOR.

If a straight line touches a circle, the straight line
drawn from the centre to the point of contact,
shall be perpendicular to the line touching the
circle.

Let the straight line DE touch the circle ABC in the point
 C ; take the centre F , and draw the straight line FC : FC is
perpendicular to DE .

For, if it be not, from the point F draw FBG perpendicu-
lar to DE ; and because FGC is a right angle, GCF is b 17. 1.
an acute angle; and to the greater
angle the greatest c side is oppo-
site: Therefore FC is greater
than FG ; but FC is equal to
 FB ; therefore FB is greater
than FG , the less than the
greater, which is impossible;
therefore FG is not perpendi-
cular to DE : In the same
manner it may be shewn, that
no other is perpendicular to it
besides FC , that is, FC is perpendicular to DE . Therefore,
a straight line, &c. Q. E. D.



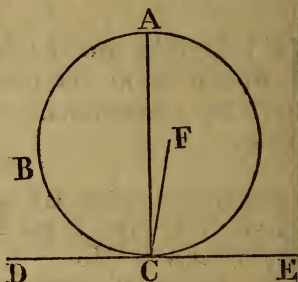
c 19. 1.

PROP. XIX. THEOR.

IF a straight line touches a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC , in C , and from C let CA be drawn at right angles to DE ; the centre of the circle is in CA .

For, if not, let F be the centre, if possible, and join CF . Because DE touches the circle ABC , and FC is drawn from the centre to the point of contact, FC is perpendicular ^a to DE ; therefore FCE is a right angle: But ACE is also a right angle; therefore the angle FCE is equal to the angle ACE , the less to the greater, which is impossible: Wherefore F is not the centre of the circle ABC : In the same manner, it may be shewn, that no other point which is not in CA , is the centre; that is, the centre is in CA . Therefore, if a straight line, &c. $Q. E. D.$



PROP. XX. THEOR.

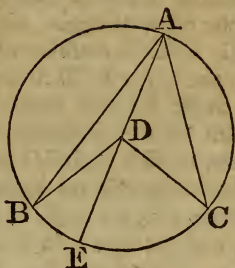
THE angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

Let ABC be a circle, and BDC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base; the angle BDC is double of the angle BAC .

First, let D , the centre of the circle, be within the angle BAC , and join AD , and produce it to E : Because DA is equal

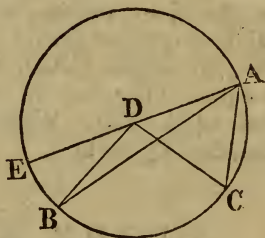
Book III.
a 5. 1.

b 32. 1.



equal to $\angle DB$, the angle $\angle DAB$ is equal ^a to the angle $\angle DBA$; therefore the angles $\angle DAB$, $\angle DBA$ are double of the angle $\angle DAB$; but the angle $\angle BDE$ is equal ^b to the angles $\angle DAB$, $\angle DBA$; therefore also the angle $\angle BDE$ is double of the angle $\angle DAB$: For the same reason, the angle $\angle EDC$ is double of the angle $\angle DAC$: Therefore the whole angle $\angle BDC$ is double of the whole angle $\angle BAC$.

Again, let D , the centre of the circle be without the angle $\angle BAC$, and join AD and produce it to E . It may be demonstrated, as in the first case, that the angle $\angle EDC$ is double of the angle $\angle DAC$, and that $\angle EDB$ a part of the first is double of $\angle DAB$ a part of the other; therefore the remaining angle $\angle BDC$ is double of the remaining angle $\angle BAC$. Therefore the angle at the centre, &c. Q. E. D.

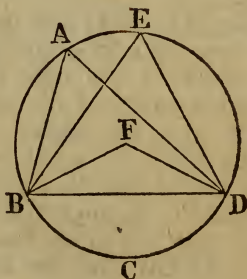


PROP. XXI. THEOR.

THE angles in the same segment of a circle are equal to one another.

Let $ABCD$ be a circle, and $\angle BAD$, $\angle BED$ angles in the same segment $BAED$: The angles $\angle BAD$, $\angle BED$ are equal to one another.

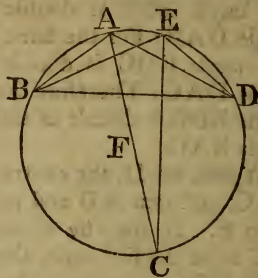
Take F the centre of the circle $ABCD$: And, first, let the segment $BAED$ be greater than a semicircle, and join BF , FD : And because the angle $\angle BFD$ is at the centre, and the angle



$\angle BAD$

Book III. BAD at the circumference, and that they have the same part of the circumference, viz. BCD, for their base; therefore the angle BFD is double ^a of the angle BAD: for the same reason, the angle BFD is double of the angle BED: Therefore the angle BAD is equal to the angle BED.

But, if the segment BAED be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to one another: Draw AF to the centre, and produce it to C, and join CE: Therefore the segment BADC is greater than a semicircle; and the angles in it BAC, BEC are equal, by the first case: For the same reason, because CBED is greater than a semicircle, the angles CAD, CED are equal: Therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the same segment, &c. Q. E. D.

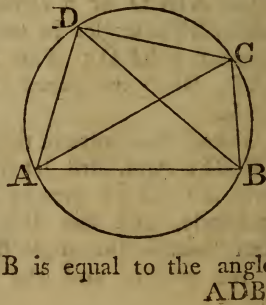


PROP. XXII. THEOR.

THE opposite angles of any quadrilateral figure described in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its opposite angles are together equal to two right angles.

Join AC, BD; and because the three angles of every triangle are equal ^a to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles: But the angle CAB is equal ^b to the angle CDB, because they are in the same segment BADC, and the angle ACB is equal to the angle



^a 32. 1.

^b 21. 3.

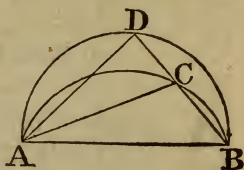
ADB,

ADB, because they are in the same segment ADCB: Book III.
 Therefore the whole angle ADC is equal to the angles CAB, ACB: To each of these equals add the angle ABC; therefore the angles ABC, CAB, BCA, are equal to the angles ABC, ADC: But ABC, CAB, BCA are equal to two right angles; therefore also the angles ABC, ADC are equal to two right angles: In the same manner, the angles BAD, DCB may be shewn to be equal to two right angles. Therefore, the opposite angles, &c. Q. E. D.

P R O P. XXIII. T H E O R.

UPON the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, let the two similar segments of circles, viz. ACB, ADB, be upon the same side of the same straight line AB, not coinciding with one another: then, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point ^a: one of the segments must therefore fall within the other: let ACB fall within ADB, and draw the straight line BCD, and join CA, DA: and because the segment ACB is similar to the segment ADB, and that similar segments of circles contain ^b equal angles; ^{b 9. def. 3.} the angle ACB is equal to the angle ADB, the exterior to the interior, which is impossible ^c. Therefore, there cannot ^{c 15. 1.} be two similar segments of a circle upon the same side of the same line, which do not coincide. Q. E. D.



a 10. 3.

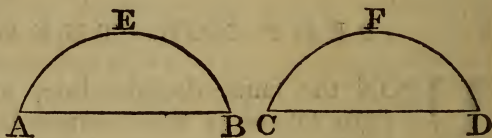
P R O P.

PROP. XXIV. THEOR.

SIMILAR segments of circles upon equal straight lines, are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD; the segment AEB is equal to the segment CFD.

For, if the segment AEB be applied to the segment CFD, so as the point A be on



C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD: Therefore the straight line AB coinciding with CD, the segment AEB must ^a coincide with the segment CFD, and therefore is equal to it. Wherefore, similar segments, &c. Q. E. D.

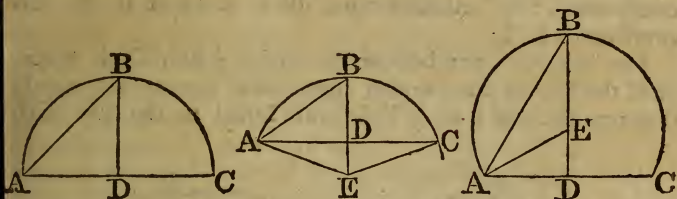
PROP. XXV. PROB.

A Segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle; it is required to describe the circle of which it is the segment.

^a 10. I. Bisect ^a AC in D, and from the point D draw ^b DB at
^b 11. I. right angles to AC, and join AB: First, let the angles ABD, BAD be equal to one another; then the straight line BD is
^c 6. I. equal ^c to DA, and therefore to DC; and because the three straight lines DA, DB, DC, are all equal; D is the centre
^d 9. 3. of the circle ^d: from the centre D, at the distance of any of the three DA, DB, DC, describe a circle; this shall pass through the other points; and the circle of which ABC is a segment is described: and because the centre D is in AC, the segment ABC is a semicircle; but if the angles ABD, BAD are not equal to one another, at the point A, in the straight

straight line AB make ^e the angle BAE equal to the angle ABD, and produce BD, if necessary, to E, and join EC: Book III.
e 23. 1.



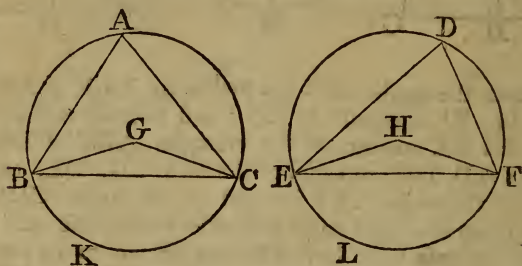
and because the angle ABE is equal to the angle BAE, the straight line BE is equal ^e to EA: and because AD is equal to DC, and DE common to the triangles ADE, CDE, the two sides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the base AE is equal ^f to the base EC: but AE was shewn to be equal ^f to EB, wherefore also BE is equal to EC: and the three straight lines AE, EB, EC are therefore equal to one another; wherefore ^d E is the centre of the circle. From the centre E, at the distance of any of the three AE, EB, EC, describe a circle, this shall pass through the other points; and the circle of which ABC is a segment is described: and it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which therefore is less than a semicircle: but if the angle ABD be less than BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle: Wherefore a segment of a circle being given, the circle is described of which it is a segment. Which was to be done.

PROP. XXVI. THEOR.

IN equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

Book III. Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences: the circumference BKC is equal to the circumference ELF .

Join BC, EF ; and because the circles ABC, DEF are equal, the straight lines drawn from their centres are equal: therefore the two sides BG, GC , are equal to the two $EH,$



HF ; and the angle at G is equal to the angle at H ; therefore the base BC is equal ^a to the base EF : and because the angle at A is equal to the angle at D , the segment BAC is ^b similar to the segment EDF ; and they are upon equal straight lines BC, EF ; but similar segments of circles upon ^c equal straight lines are equal to one another, therefore the segment BAC is equal to the segment EDF : but the whole circle ABC is equal to the whole DEF ; therefore the remaining segment BKC is equal to the remaining segment ELF , and the circumference BKC to the circumference ELF . Wherefore, in equal circles, &c. **Q. E. D.**

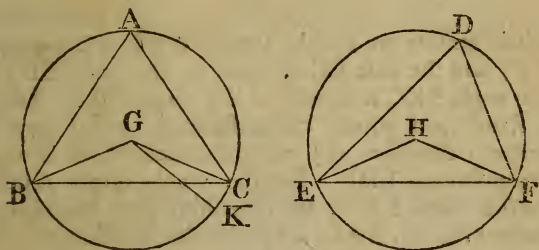
PROP. XXVII. THEOR.

IN equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF stand upon the equal circumferences BC, EF : the angle BGC is equal

equal to the angle EHF, and the angle BAC to the angle EDF. Book III

If the angle BGC be equal to the angle EHF, it is manifest^a that the angle BAC is also equal to EDF. But, if not, a 20. 3. one of them is the greater: let BGC be the greater, and at the point G, in the straight line BG, make^b the angle BGK b 23. 1. equal to the angle EHF; but equal angles stand upon equal circumferences^c, when they are at the centre; therefore the c 26. 3. circumference BK is equal to the circumference EF: but EF is equal to BC; therefore also BK is equal to BC, the less to the greater, which is impossible: therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it:



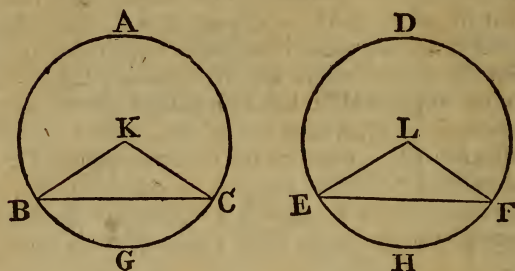
and the angle at A is half of the angle BGC, and the angle at D half of the angle EHF: therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVIII. THEOR.

IN equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater circumferences BAC, EDF, and the two less BGC, EHF: the greater BAC is equal to the greater EDF, and the less BGC to the less EHF.

Book III. Take ^a K, L, the centres of the circles, and join BK, KC
^a 1. 3. EL, LF : and because the circles are equal, the straight line



from their centres are equal ; therefore BK, KC are equal to
^b 8. 1. EL, LF ; and the base BC is equal to the base EF ; there-
^c 26. 3. fore the angle BKC is equal ^b to the angle ELF : but equal
 angles stand upon equal ^c circumferences, when they are at
 the centres ; therefore the circumference BGC is equal to
 the circumference EHF. But the whole circle ABC is equal
 to the whole EDF ; the remaining part therefore of the
 circumference, viz. BAC, is equal to the remaining part EDF.
 Therefore, in equal circles, &c. Q. E. D.

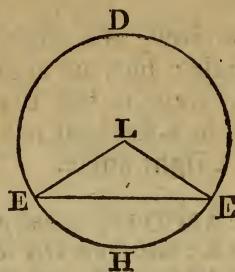
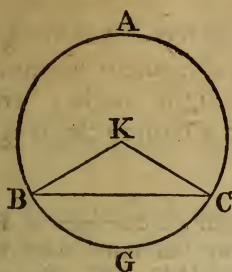
PROP. XXIX. THEOR.

IN equal circles equal circumferences are subtended
 ed by equal straight lines.

Let ABC, DEF be equal circles, and let the circumferences
 BGC, EHF also be equal ; and join BC, EF : the straight
 line BC is equal to the straight line EF.

^a 1. 3. Take ^a K, L the centres of the circles, and join BK, KC
^b 27. 3. EL, LF : and because the circumference BGC is equal to the
 circumference EHF, the angle BKC is equal ^b to the angle
 ELF : and because the circles ABC, DEF are equal, the
 straight lines from their centres are equal : therefore BK, KC
 are

are equal to EL, LF, and they contain equal angles : there- Book III.



ore the base BC is equal ^c to the base EF. Therefore, in e- c 4. 1.
ual circles, &c. Q. E. D.

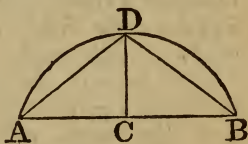
P R O P. XXX. P R O B.

TO bisect a given circumference, that is, to divide it into two equal parts.

Let ADB be the given circumference ; it is required to bi-
sect it.

Join AB, and bisect ^a it in C ; from the point C draw CD ^{a 10. 1.}
right angles to AB, and join AD, DB : the circumference
DB is bisected in the point D.

Because AC is equal to CB, and CD common to the tri-
gles ACD, BCD, the two sides
C, CD are equal to the two BC,
D ; and the angle ACD is equal
the angle BCD, because each of
em is a right angle ; therefore the
se AD is equal ^b to the base BD.



at equal straight lines cut off equal ^c circumferences, the
reater equal to the greater, and the less to the less, and AD,
B are each of them less than a semicircle ; because DC
ies through the centre ^d : Wherefore the circumference ^{d Cor. 1. 3.}
D is equal to the circumference DB : therefore the given
umference is bisected in D. Which was to be done.

H

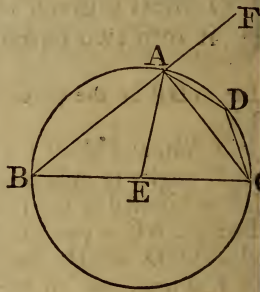
P R O P.

PROP. XXXI. THEOR.

IN a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let $ABCD$ be a circle, of which the diameter is BC , and centre E ; and draw CA dividing the circle into the segments ABC , ADC , and join BA , AD , DC ; the angle in the semicircle BAC is a right angle; and the angle in the segment ABC , which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC , which is less than a semicircle, is greater than a right angle.

Join AE , and produce BA to F ; and because BE is equal ^a to EA , the angle EAB is equal ^a to EBA ; also, because AE is equal to EC the angle EAC is equal to ECA ; wherefore the whole angle BAC is equal to the two angles ABC , ACB . But FAC , the exterior angle of the triangle ABC , is also equal ^b to the two angles ABC , ACB ; therefore the angle BAC is equal to the angle FAC , and each of them is therefore a right ^c angle: wherefore the angle BAC in a semicircle is a right angle.



And because the two angles ABC , BAC of the triangle ABC are together less ^d than two right angles, and that BAC is a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC greater than a semicircle, is less than a right angle.

And because $ABCD$ is a quadrilateral figure in a circle, any two of its opposite angles are equal ^e to two right angles; therefore the angles ABC , ADC are equal to two right angles; and ABC is less than a right angle; wherefore the other ADC is greater than a right angle.

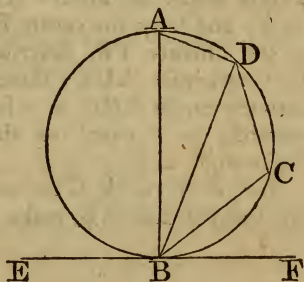
COR. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two; and when the adjacent angles are equal, they are right angles.

PROP. XXXII. THEOR.

IF a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let the straight line EF touch the circle $ABCD$ in B , and from the point B let the straight line BD be drawn cutting the circle: The angles which BD makes with the touching line EF shall be equal to the angles in the alternate segments of the circle: that is, the angle FBD is equal to the angle which is in the segment DAB , and the angle DBE to the angle in the segment BCD .

From the point B draw a BA at right angles to EF , and take any point C in the circumference BD , and join AD , DC , CB ; and because the straight line EF touches the circle $ABCD$ in the point B , and BA is drawn at right angles to the touching line from the point of contact B , the centre of the circle is b in BA ; therefore the angle ADB in a semicircle is a right c angle, and consequently the other two angles BAD , ABD are equal d to a right angle: but ABF is likewise a right angle; therefore the angle ABF is equal to the angles BAD , ABD : take



a 11. 1.

b 19. 3.

c 31. 3.

d 32. 1.

H 2

from

Book III. from these equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is in the alternate segment of the circle: and because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are equal ^e to two right angles; therefore the angles DBF, DBE, being likewise equal ^f to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD: therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle.

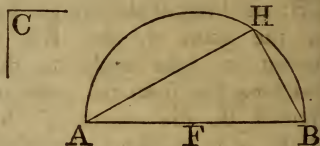
Wherefore, if a straight line, &c. Q. E. D.

PROP. XXXIII. PROB.

UPON a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and the angle at C the given rectilineal angle; it is required to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.

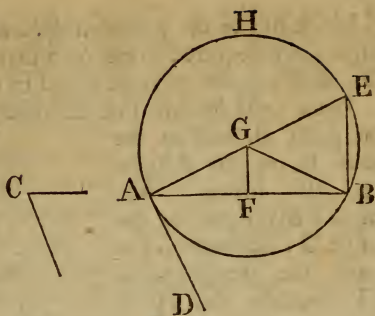
First, let the angle at C be a right angle, and bisect ^a AB in F, and from the centre F, at the distance FB, describe the semicircle AHB; therefore the angle AHB in a semicircle is ^b equal to the right angle at C.



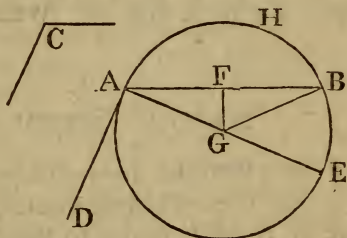
But, if the angle C be not a right angle, at the point A, in the straight line AB, make ^c the angle BAD equal to the angle

Book III.
d 11. 1.

angle C, and from the point A draw d AE at right angles to AD; bisect a AB in F, and from F draw d FG at right angles to AB, and join GB: And because AF is equal to FB, and FG common to the triangles AFG, BFG the two sides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal e to the base e 4. 1. GB; and the circle described from the centre G, at the distance GA, shall pass through the point B; let this be the circle AHB: And because from the point A the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD f touches the circle; and because AB drawn f Cor. 16. 3.



from the point of contact A cuts the circle, the angle CAB is equal to the angle in the alternate segment AHB g: but the angle CAB is equal to the angle in the alternate segment AHB: Wherefore, upon the given straight line AB the segment AHB



g 32. 3.

a circle is described which contains an angle equal to the given angle at C. Which was to be done.

PROP. XXXIV. PROB.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilinear angle.

H 3

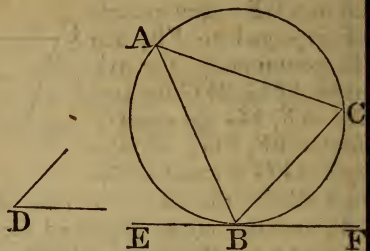
Let

Book III. Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a segment from the circle ABC that shall contain an angle equal to the angle D .

a 17. 3. Draw ^a the straight line EF touching the circle ABC in the point B , and at the point B , in the straight

b 23. 1. line BF make ^b the angle FBC equal to the angle D ; therefore, because the straight line EF touches the circle ABC , and BC is drawn from the point of contact B , the angle FBC is

c 32. 3. equal ^c to the angle in the alternate segment BAC of the circle: but the angle FBC is equal to the angle D ; therefore the angle in the segment BAC is equal to the angle D : wherefore the segment BAC is cut off from the given circle ABC containing an angle equal to the given angle D . Which was to be done.

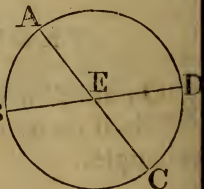


PROP. XXXV. THEOR.

IF two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

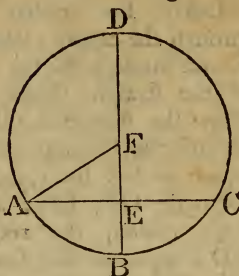
Let the two straight lines AC , BD , within the circle $ABCD$, cut one another in the point E : the rectangle contained by AE , EC is equal to the rectangle contained by BE , ED .

If AC , BD pass each of them through the centre, so that E is the centre; it is evident, that AE , EC , BE , ED , being ^B all equal, the rectangle AE , EC is likewise equal to the rectangle BE , ED .



But let one of them BD pass through the centre, and cut the other AC , which does not pass through the centre, at right

right angles, in the point E : then, if BD be bisected in F, F is the centre of the circle ABCD ; join AF : and because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E, AE, EC are equal ^a to one another : and because the straight line BD is cut into two equal parts in the point F, and into two unequal in the point E, the rectangle BE, ED, together with the square of EF, is equal ^b to the square of FB ; that is, to the square of FA ; but the squares of AE, EF are equal ^c to the square of FA ; therefore the rectangle BE, ED, together with the square of EF, is equal to the squares of AE, EF : take away the common square of EF, and the remaining rectangle BE, ED is equal to the remaining square of AE ; that is, to the rectangle AE, EC.

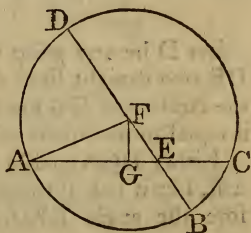


a 3. 3.

b 5. 2.

c 47. 1.

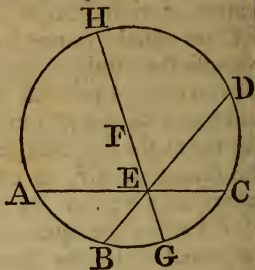
Next, Let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles : then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and from F draw ^d FG perpendicular to AC ; therefore AG is equal ^a to GC ; wherefore, the rectangle AE, EC, together with the square of EG, is equal ^b to the square of AG : to each of these equals add the square of GF ; therefore the rectangle AE, EC, together with the squares of EG, GF, is equal to the squares of AG, GF : But the squares of EG, GF are equal ^c to the square of EF ; and the squares of AG, GF are equal to the square of AF : therefore the rectangle AE, EC, together with the square of EF, is equal to the square of AF ; that is, to the square of FB : but the square of FB is equal ^b to the rectangle BE, ED, together with the square of EF ; therefore the rectangle AE, EC, together with the square of EF, is equal to the rectangle BE, ED, together with the square of EF : take a-



d 12. 1.

Book III. way the common square of EF , and the remaining rectangle AE, EC , is therefore equal to the remaining rectangle BE, ED .

Lastly, Let neither of the straight lines AC, BD pass through the centre: take the centre F , and through E , the intersection of the straight lines AC, DB , draw the diameter $GEFH$: and because the rectangle AE, EC is equal, as has been shown, to the rectangle GE, EH ; and, for the same reason, the rectangle BE, ED is equal to the same rectangle GE, EH ; therefore the rectangle AE, EC is equal to the rectangle BE, ED . Wherefore, if two straight lines, &c. Q. E. D.



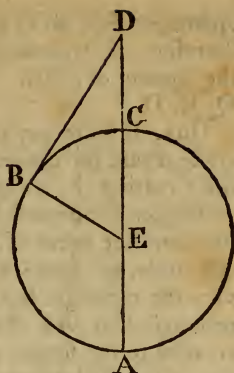
PROP. XXXVI. THEOR.

IF from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

Let D be any point without the circle ABC , and DCA, DB two straight lines drawn from it, of which DCA cuts the circle, and DB touches the same: the rectangle AD, DC is equal to the square of DB .

Either DCA passes through the centre, or it does not; first, Let it pass through the centre E , and join EB ; therefore the angle EBD is a right ^a angle: and because the straight

straight line AC is bisected in E, and produced to the point D, the rectangle AD, DC, together with the square of EC, is equal ^b to the square of ED; and CE is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED: but the square of ED is equal ^c to the squares of EB, BD, because EBD is a right angle: therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD: take away the common square of EB; therefore the remaining rectangle AD, DC is equal to the square of the tangent DB.



b 6. 2.

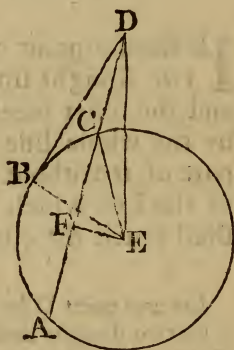
c 47. 34

But if DCA does not pass through the centre of the circle ABC, take ^d the centre E, and draw EF perpendicular ^e to AC, and join EB, EC, ED: and because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the

d 1. 3.

e 12. 1.

centre, at right angles, it shall likewise bisect ^f it; therefore AF is equal to FC: and because the straight line AC is bisected in F, and produced to D, the rectangle AD, DC, together with the square of FC, is equal ^b to the square of FD: to each of these equals add the square of FE; therefore the rectangle AD, DC, together with the squares of CF, FE, is equal to the squares of DF, FE: but the square of ED is equal ^c to the squares of DF, FE be-

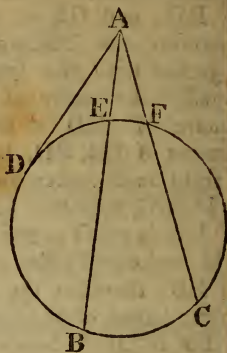


f 3. 5.

cause EFD is a right angle; and the square of EC is equal to the squares of CF, FE; therefore the rectangle AD, DC, together with the square of EC, is equal to the square of ED: and CE is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED: but the squares of EB, BD are equal to the square ^c of ED, because EBD is a right angle; therefore the rectangle AD, DC, together with the square of EB, is equal to the

Book III. squares of EB, BD : take away the common square of EB therefore the remaining rectangle AD, DC is equal to the square of DB . Wherefore, if from any point, &c Q. E. D.

COR. If from any point without a circle, there be drawn two straight lines cutting it, as AB, AC , the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF : for each of them is equal to the square of the straight line AD which touches the circle.



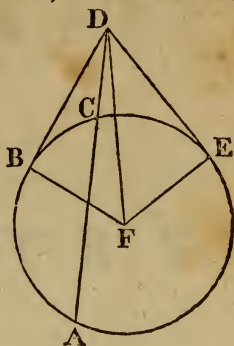
PROP. XXXVII. THEOR.

IF from a point without a circle there be drawn two straight lines, one of which cuts the circle and the other meets it ; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the square of the line which meets it, the line which meets shall touch the circle.

Let any point D be taken without the circle ABC , and from it let two straight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it ; if the rectangle AD, DC , be equal to the square of DB ; DB touches the circle.

- a 17. 3. Draw ^a the straight line DE touching the circle ABC find the centre F , and join FE, FB, FD ; then FED is
 b 18. 3. right ^b angle : and because DE touches the circle ABC , and
 c 36. 3. DCA cuts it, the rectangle AD, DC is equal ^c to the square of DE ; but the rectangle AD, DC is, by hypothesis, equal to the square of DB : therefore the square of DE is equal to the square of DB ; and the straight line DE equal to the straight

straight line DB: and FE is equal to FB, wherefore DE, Book III.
 EF are equal to DB, BF; and the
 side FD is common to the two
 triangles DEF, DBF; therefore
 the angle DEF is equal ^d to the
 angle DBF; and DEF is a right
 angle, therefore also DBF is a right
 angle: but FB, if produced, is a
 diameter, and the straight line
 which is drawn at right angles to a
 diameter, from the extremity of it,
 touches ^e the circle: therefore DB
 touches the circle ABC. Where-
 fore, if from a point, &c. Q. E. D.



d 8. 1.

e 16 3.

E L E.

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E L E M E N T S

O F

G E O M E T R Y.

B O O K I V.

D E F I N I T I O N S.

I.

A Rectilineal figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each. Book IV.

II.

In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.



III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.



IV.

Book IV.

IV.

A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.



VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



VII.

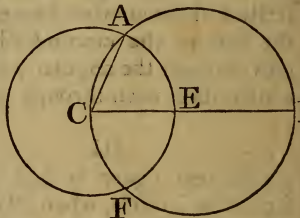
A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

IN a given circle to place a straight line, equal to a given straight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a straight line BC is placed equal to D: But, if it is not, BC is greater than D; make CE equal^a to D, and from the centre C, at the distance CE, describe the circle



a 3. 1.

AEF, and join CA: Therefore, because C is the centre of

th

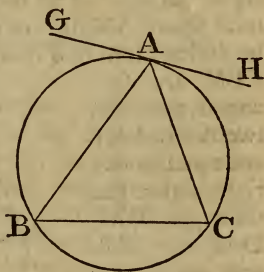
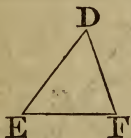
circle AEF, CA is equal to CE; but D is equal to CE; Book IV.
 therefore D is equal to CA: Wherefore, in the circle ABC,
 a straight line is placed, equal to the given straight line D,
 which is not greater than the diameter of the circle. Which
 was to be done.

PROP. II. PROB.

IN a given circle to inscribe a triangle equiangu-
 lar to a given triangle.

Let ABC be the given circle, and DEF the given tri-
 angle; it is required to inscribe in the circle ABC a triangle
 equiangular to the triangle DEF.

Draw ^a the straight line GAH touching the circle in the ^{a 17. 3.}
 point A, and at the point A, in the straight line AH, make ^b ^{b 23. 1.}
 the angle HAC equal to the angle DEF; and at the point A,
 in the straight line
 AG, make the angle
 GAB equal to the
 angle DFE, and join
 BC: Therefore, be-
 cause HAG touches
 the circle ABC, and
 AC is drawn from
 the point of contact,
 the angle HAC is e-
 qual ^c to the angle



^c 32. 3.

ABC in the alternate segment of the circle: But HAC is e-
 qual to the angle DEF; therefore also the angle ABC is e-
 qual to DEF: for the same reason, the angle ACB is equal
 to the angle DFE; therefore the remaining angle BAC is
 equal ^d to the remaining angle EDF: Wherefore the triangle ^{d 32. 1.}
 ABC is equiangular to the triangle DEF, and it is inscribed
 in the circle ABC. Which was to be done.

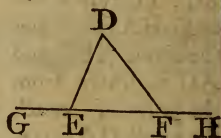
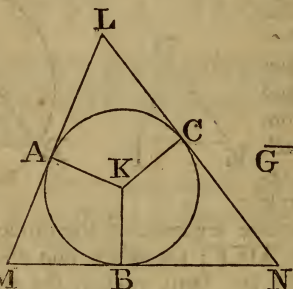
PROP.

PROP. III. PROB.

ABOUT a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to describe a triangle about the circle ABC equiangular to the triangle DEF .

Produce EF both ways to the points G, H , and find the centre K of the circle ABC , and from it draw any straight line KB ; at the point K in the straight line KB , make ^a the angle BKA equal to the angle DEG , and the angle BKC equal to the angle DFH ; and through the points A, B, C draw the straight lines LAM, MBN, NCL touching ^b the circle ABC : Therefore, because LM, MN, NL touch the circle ABC in the points A, B, C , to which from the centre are drawn KA, KB, KC , the angles at the points A, B, C , are right ^c angles. And because the four angles of the quadrilateral figure $AMBK$ are equal to four right angles, for it can be divided into two triangles, and because two of them, KAM, KBM are right angles, the other two AKB, AMB are equal to two right angles: But the angles DEG, DEF are like-
^d wise equal ^d to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF , of which AKB is equal to DEG ; wherefore the remaining angle AMB is equal to the remaining angle DEF : In like manner, the angle LMN may be demonstrated to be equal to DFE ; and therefore the remaining angle MLN is equal ^e to the remaining angle EDF : Wherefore the triangle LMN is equiangular to the triangle DEF : And it is described about the circle ABC . Which was to be done.



PROP.

PROP. IV. PROB.

TO inscribe a circle in a given triangle.

Let the given triangle be ABC ; it is required to inscribe a circle in ABC .

Bisect^a the angles ABC , BCA by the straight lines BD , ^{a 9. 1.}
 CD meeting one another in the point D , from which draw^b ^{b 12. 1.}

DE , DF , DG perpendiculars

to AB , BC , CA : And be-

cause the angle EBD is equal

to the angle FBD , the angle

ABC being bisected by BD ;

and because the right angle

ED is equal to the right angle

FD , the two triangles EBD ,

BD have two angles of the

one equal to two angles of the

other; and the side BD , which

is opposite to one of the equal

angles in each, is common to both; therefore their other sides

are equal^c; wherefore DE is equal to DF . For the same ^{c 26. 1.}

reason, DG is equal to DF ; therefore the three straight lines

DE , DF , DG are equal to one another, and the circle de-

scribed from the centre D , at the distance of any of them, will

pass through the extremities of the other two, and will touch the

straight lines AB , BC , CA , because the angles at the points

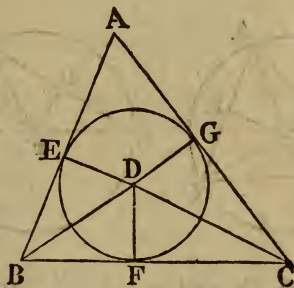
E , F , G are right angles, and the straight line which is drawn

from the extremity of a diameter at right angles to it, touches

the circle: Therefore the straight lines AB , BC , CA , do ^{d 16. 3.}

each of them touch the circle, and the circle EFG is inscribed

in the triangle ABC . Which was to be done.



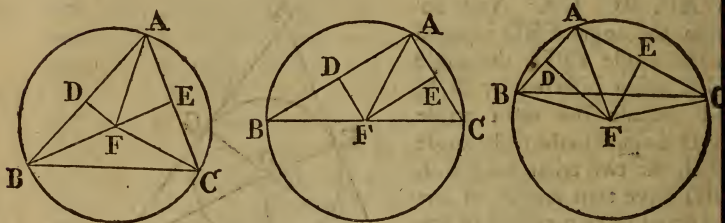
PROP.

PROP. V. PROB.

TO describe a circle about a given triangle.

Let the given triangle be ABC ; it is required to describe a circle about ABC .

a 10. 1. Bisect^a AB , AC in the points D , E , and from these points draw DF , EF at right angles^b to AB , AC ; DF , EF produce



meet one another; for, if they do not meet, they are parallel wherefore AB , AC , which are at right angles to them, are parallel, which is absurd: Let them meet in F , and join FA also, if the point F be not in BC , join BF , CF : then, because AD is equal to DB , and DF common, and at right angles to AB , the base AF is equal^c to the base FB . In like manner it may be shown that CF is equal to FA ; and therefore BF is equal to FC ; and FA , FB , FC are equal to one another wherefore the circle, described from the centre F , at the distance of one of them, shall pass through the extremities of the other two, and be described about the triangle ABC , which was to be done.

COR. And it is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but, when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and, if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle.

Wherefore

Wherefore, if the given triangle be acute angled, the centre of the circle falls within it; if it be a right angled triangle, the centre is in the side opposite to the right angle; and, if it be an obtuse angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

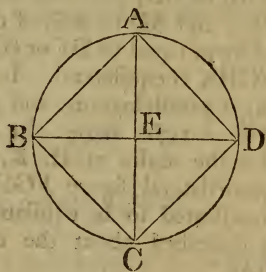
PROP. VI. PROB.

TO inscribe a square in a given circle.

Let ABCD be the given circle; it is required to inscribe a square in ABCD.

Draw the diameters AC, BD at right angles to one another, and join AB, BC, CD, DA; because BE is equal to

ED, E being the centre, and because EA is at right angles to BD, and common to the triangles ABE, ADE; the base BA is equal^a to the base AD; and, for the same reason, BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is also rectangular; for the straight line BD, being the diameter of the circle ABCD, BAD is a



^a 4. I.

semicircle; wherefore the angle BAD is a right^b angle; for the same reason each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been shown to be equilateral; therefore it is a square; and it is inscribed in the circle ABCD. Which was to be done.

PROP. VII. PROB.

TO describe a square about a given circle.

Let ABCD be the given circle; it is required to describe a square about it.

I 2

Draw

Book IV.

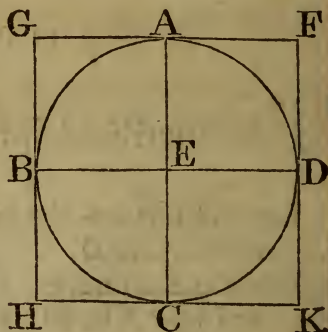
a 17. 3.

b 18. 3.

c 28. 1.

d 34. 1.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D draw a FG, GH, HK, KF touching the circle; and because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right angles; for the same reason, the angles at the points B, C, D are right angles; and because the angle AEB is a right angle, as likewise is EBG, GH is parallel to AC; for the same reason, AC is parallel to FK, and in like manner GF, HK may each of them be demonstrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and GF is therefore equal to HK, and GH to FK; and because AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK: GH, FK are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is also rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB is likewise a right angle: In the same manner, it may be shown that the angles at H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular; and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle ABCD. Which was to be done.



PROP. VIII. PROB.

TO inscribe a circle in a given square.

Let ABCD be the given square; it is required to inscribe a circle in ABCD.

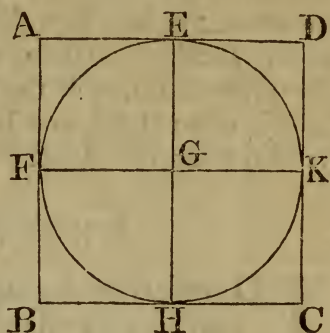
a 10. 1.

b 31. 1.

c 34. 1.

Bisect each of the sides AB, AD, in the points F, E, and through E draw EH parallel to AB or DC, and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram and their opposite sides are equal; and because AD is equal

to AB, and that AE is the half of AD, and AF the half of Book IV.
 AB, AE is equal to AF; wherefore the sides opposite to
 these are equal, viz. FG to GE; in the same manner, it may
 be demonstrated, that GH, GK
 are each of them equal to FG or
 GE; therefore the four straight
 lines GE, GF, GH, GK, are e-
 qual to one another; and the cir-
 cle described from the centre G,
 at the distance of one of them,
 shall pass through the extremities
 of the other three; and shall also
 touch the straight lines AB, BC,
 CD, DA, because the angles at
 the points E, F, H, K are right d
 angles, and because the straight line which is drawn from the
 extremity of a diameter, at right angles to it, touches the
 circle; therefore each of the straight lines AB, BC, CD, DA e 16. 3.
 touches the circle, which therefore is inscribed in the square
 ABCD. Which was to be done.



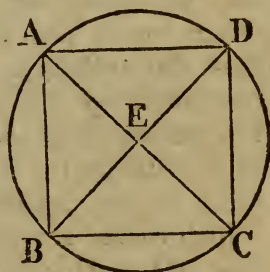
d 29. 1.

PROP. IX. PROB.

To describe a circle about a given square.

Let ABCD be the given square; it is required to describe
 circle about it.

Join AC, BD, cutting one another in E; and because DA
 is equal to AB, and AC common to the triangles DAC,
 BAC, the two sides DA, AC are equal to the two BA, AC,
 and the base DC is equal to the base
 BC; wherefore the angle DAC is equal
 to the angle BAC, and the angle
 DAB is bisected by the straight line
 AC: In the same manner, it may be
 demonstrated, that the angles ABC,
 DCB, CDA are severally bisected by
 the straight lines BD, AC; therefore,
 because the angle DAB is equal to the
 angle ABC, and that the angle EAB is the half of DAB,



a 8. 1.

Book IV.
b 6. 1.

and EBA the half of ABC ; the angle EAB is equal to the angle EBA ; wherefore the side EA is equal^b to the side EB : In the same manner, it may be demonstrated, that the straight lines EC , ED are each of them equal to EA or EB : therefore the four straight lines EA , EB , EC , ED are equal to one another; and the circle described from the centre E , at the distance of one of them, shall pass through the extremities of the other three, and be described about the square $ABCD$. Which was to be done.

PROP. X. PROB.

TO describe an isosceles triangle, having each of the angles at the base double of the third angle.

a 11. 2.

Take any straight line AB , and divide^a it in the point C , so that the rectangle AB , BC may be equal to the square of CA

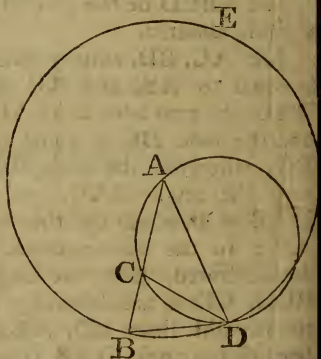
b 1. 4.

and from the centre A , at the distance AB , describe the circle BDE , in which place^b the straight line BD equal to AC

c 5. 4.

which is not greater than the diameter of the circle BDE . join DA , DC , and about the triangle ADC describe^c the circle ACD ; the triangle ABD is such as is required, that is each of the angles ABD , ADB is double of the angle BAD .

Because the rectangle AB , BC is equal to the square of AC , and that AC is equal to BD , the rectangle AB , BC is equal to the square of BD ; and because from the point B without the circle ACD two straight lines BCA , BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB , BC contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD which meets it; the straight line BD touches^d the circle ACD ; and because BD touches the circle, and DC is drawn



d 37. 3.

drawn from the point of contact D, the angle BDC is equal ^e to the angle DAC in the alternate segment of the circle; to each of these add the angle CDA; therefore the whole angle BDA is equal to the two angles CDA, DAC; but the exterior angle BCD is equal ^f to the angles CDA, DAC; therefore also BDA is equal to BCD; but BDA is equal ^g to CBD, because the side AD is equal to the side AB; therefore CBD, or DBA is equal to BCD; and consequently the three angles BDA, DBA, BCD, are equal to one another: And because the angle DBC is equal to the angle BCD, the side BD is equal ^h to the side DC; but BD was made equal to CA; therefore also CA is equal to CD, and the angle CDA equal ^g to the angle DAC; therefore the angles CDA, DAC together, are double of the angle DAC: but BCD is equal to the angles CDA, DAC; therefore also BCD is double of DAC; and BCD is equal to each of the angles BDA, DBA, each therefore of the angles BDA, DBA is double of the angle DAB; wherefore an isosceles triangle ABD is described, having each of the angles at the base double of the third angle. Which was to be done.

Book IV.

c 32. 3.

f 32. 1.

g 5. 1.

h 6. 1.

PROP. XI. PROB.

TO inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle, it is required to inscribe an equilateral and equiangular pentagon in the circle ABCDE.

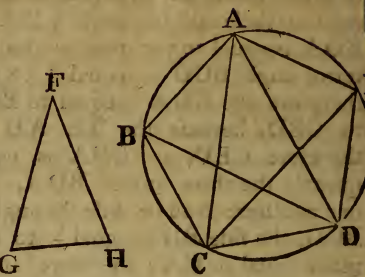
Describe ^a an isosceles triangle FGH, having each of the angles at G, H, double of the angle at F; and in the circle ABCDE inscribe ^b the triangle ACD equiangular to the triangle

a 10. 4.

b 2. 4.

I 4

triangle

Book IV. triangle FGH, so that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. c 9. 1. Bisect ^c the angles ACD, CDA by the straight lines CE, DB; and join AB, BC, DE, EA. ABCDE is the pentagon required. 

Because angles the ACD, CDA are each of them double of CAD, and are bisected by the straight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but equal angles stand upon equal ^d circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another: and equal circumferences are subtended by equal ^e straight lines; therefore the five straight lines AE, BC, CD, DE, EA are equal to one another. Wherefore the pentagon ABCDE is equilateral. It is also equiangular; because the circumference AB is equal to the circumference DE: if to each be added BCD, the whole ABCD is equal to the whole EDCB: and the angle AED stands on the circumference ABCD, and the angle BAE on the circumference EDCB; therefore the angle BAE is equal ^f to the angle AED: for the same reason, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED: therefore the pentagon ABCDE is equiangular; and it has been shown that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed. Which was to be done.

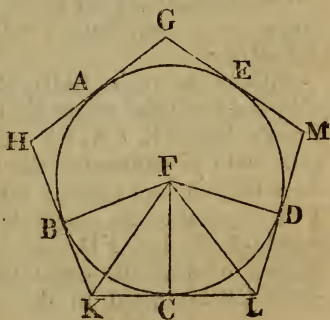
PROP. XII. PROB.

TO describe an equilateral and equiangular pentagon about a given circle.

Let

Let $ABCDE$ be the given circle, it is required to describe Book IV.
an equilateral and equiangular pentagon about the circle $ABCDE$.

Let the angles of a pentagon, inscribed in the circle, by the
last proposition, be in the points A, B, C, D, E , so that the
circumferences AB, BC, CD, DE, EA are equal ^a 11. 4.
and through the points A, B, C, D, E draw $GH, HK, KL, LM,$
 MG , touching ^b the circle: take the centre F , and join $FB,$ 17. 3.
 FK, FC, FL, FD : and because the straight line KL touches
the circle $ABCDE$ in the point C , to which FC is drawn
from the centre F , FC is perpendicular ^c to KL ; therefore 18. 3.
each of the angles at C is a right angle: for the same reason,
the angles at the points B, D are right angles; and because
 $\angle FCK$ is a right angle, the square of FK is equal ^d to the 47. 1.
squares of FC, CK . For the same reason, the square of FK is
equal to the squares of FB, BK : therefore the squares of $FC,$
 CK are equal to the squares of FB, BK , of which the square
of FC is equal to the square of FB ; the remaining square of
 CK is therefore equal to the remaining square of BK , and the
straight line CK equal to BK : and because FB is equal to FC ,
and FK common to the triangles BFK, CFK , the two $BF,$
 FK are equal to the two CF, FK ; and the base BK is equal
to the base KC ; therefore the angle BFK is equal ^e to the 8. 1.
angle KFC , and the angle BKF to FKC ; wherefore the
angle BFC is double of the angle KFC , and BKC double of
 KFC : for the same reason, the angle CFD is double of the
angle CFL , and CLD double of CLF : and because the cir-
cumference BC is equal to the circumference CD , the angle
 BFC is equal ^f to the angle CFD ; and BFC is double of 27. 3
the angle KFC , and CFD
double of CFL ; therefore
the angle KFC is equal to
the angle CFL ; and the
right angle FCK is equal to
the right angle FCL : there-
fore in the two triangles
 BKC, FLC , there are two
angles of one equal to two
angles of the other, each to
each, and the side FC , which
is adjacent to the equal angles



in

Book IV. in each, is common to both; therefore the other sides shall be
 g 26. I. equal g to the other sides, and the third angle to the third
 angle: therefore the straight line KC is equal to CL, and the
 angle FKC to the angle FLC: and because KC is equal to
 CL, KL is double of KC: in the same manner, it may be
 shown that HK is double of BK: and because BK is equal to
 KC, as was demonstrated, and that KL is double of KC, and
 HK double of BK, HK shall be equal to KL: in like man-
 ner, it may be shown that GH, GM, ML are each of them
 equal to HK or KL: therefore the pentagon GHKLM is
 equilateral. It is also equiangular; for, since the angle FKC
 is equal to the angle FLC, and the angle HKL double
 of the angle FKC, and KLM double of FLC, as was before
 demonstrated, the angle HKL is equal to KLM: and in like
 manner it may be shown, that each of the angles KHG, HGM,
 GML is equal to the angle HKL or KLM: therefore the five
 angles GHK, HKL, KLM, LMG, MGH being equal to one
 another, the pentagon GHKLM is equiangular: and it is equi-
 lateral, as was demonstrated; and it is described about the
 circle ABCDE. Which was to be done.

PROP. XIII. PROB.

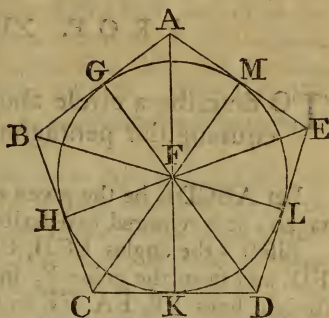
TO inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inscribe a circle in the pentagon ABCDE.

a 9. I. Bisect a the angles BCD, CDE by the straight lines CF, DF, and from the point F, in which they meet, draw the straight lines FB, FA, FE: therefore, since BC is equal to CD, and CF common to the triangles BCF, DCF, the two sides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF; therefore the base BF is equal b to the base FD, and the other angles to the other angles, to which the equal sides are opposite; therefore the angle CBF is equal to the angle CDF: and because the
 b 4. I.

angle CDE is double of CDF, and that CDE is equal to Book IV.
CBA, and CDF to CBF;

CBA is also double of the
angle CBF; therefore the
angle ABF is equal to the
angle CBF; wherefore the
angle ABC is bisected by the
straight line BF: In the same
manner, it may be demon-
strated, that the angles BAE,
AED, are bisected by the
straight lines AF, FE: from
the point F draw ϵ FG, FH,
FK, FL, FM perpendiculars
to the straight lines AB, BC,
CD, DE, EA: and because the angle HCF is equal to KCF,



C 12. I.

and the right angle FHC equal to the right angle FKC; in the
triangles FHC, FKC there are two angles of one equal to
two angles of the other, and the side FC, which is opposite to
one of the equal angles in each, is common to both; there-
fore the others sides shall be equal d , each to each; wherefore d 26. I.
the perpendicular FH is equal to the perpendicular FK: in
the same manner it may be demonstrated, that FL, FM, FG
are each of them equal to FH or FK: therefore the five
straight lines FG, FH, FK, FL, FM are equal to one ano-
ther: wherefore the circle described from the centre F, at
the distance of one of these five, shall pass through the ex-
tremities of the other four, and touch the straight lines AB,
BC, CD, DE, EA, because that the angles at the points G, H,
K, L, M are right angles, and that a straight line drawn from
the extremity of the diameter of a circle at right angles to
it, touches ϵ the circle: therefore each of the straight lines
AB, BC, CD, DE, EA touches the circle; wherefore it is
inscribed in the pentagon ABCDE. Which was to be
demonstrated.

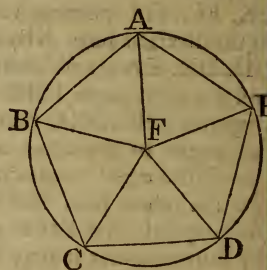
P R O P.

PROP. XIV. PROB.

TO describe a circle about a given equilateral and equiangular pentagon.

Let $ABCDE$ be the given equilateral and equiangular pentagon; it is required to describe a circle about it.

a 9. 1. Bisect the angles BCD , CDE by the straight lines CF , FD , and from the point F , in which they meet, draw the straight lines FB , FA , FE to the points B , A , E . It may be demonstrated, in the same manner as in the preceding proposition, that the angles CBA , BAE , AED are bisected by the straight lines FB , FA , FE : and because the angle BCD is equal to the angle CDE , and that FCD is the half of the angle BCD , and CDF the half of CDE ; the angle



b 6. 1. FCD is equal to FDC ; wherefore the side CF is equal b t the side FD : In like manner it may be demonstrated, that FB , FA , FE are each of them equal to FC or FD : therefore the five straight lines FA , FB , FC , FD , FE are equal to one another; and the circle described from the centre F , at the distance of one of them, shall pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon $ABCDE$. Which was to be done.

PROP. XV. PROB.

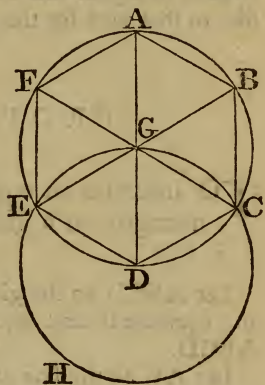
TO inscribe an equilateral and equiangular hexagon in a given circle.

Let $ABCDEF$ be the given circle; it is required to inscribe Book IV.
an equilateral and equiangular hexagon in it.

Find the centre G of the circle $ABCDEF$, and draw the diameter AGD ; and from D as a centre, at the distance DG , describe the circle $EGCH$, join EG , CG , and produce them to the points B , F ; and join AB , BC , CD , DE , EF , FA : the hexagon $ABCDEF$ is equilateral and equiangular.

Because G is the centre of the circle $ABCDEF$, GE is equal to GD : and because D is the centre of the circle $EGCH$, DE is equal to DG ; wherefore GE is equal to ED , and the triangle EGD is equilateral; and therefore its three angles EGD , GDE , DEG are equal to one another^a; and ^a Cor. 5. 1. the three angles of a triangle are equal^b to two right angles; ^b 32. 1. wherefore the angle EGD is the third part of two right angles:

In the same manner it may be demonstrated that the angle DGC is also the third part of two right angles: and because the straight line GC makes with EB the adjacent angles EGC , CGB equal^c to two right angles; the remaining angle CGB is the third part of two right angles; therefore the angles EGD , DGC , CGB , are equal to one another: and to these are equal^d the vertical opposite angles BGA , AGF , FGE ; therefore the six angles EGD , DGC , CGB , BGA , AGF , FGE are equal to one another. But equal



angles stand upon equal^e circumferences; therefore the six ^e 26. 3. circumferences AB , BC , CD , DE , EF , FA are equal to one another: and equal circumferences are subtended by equal^f ^f 29. 3. straight lines; therefore the six straight lines are equal to one another, and the hexagon $ABCDEF$ is equilateral. It is also equiangular; for, since the circumference AF is equal to ED , to each of these add the circumference $ABCD$; therefore the whole circumference $FABCD$ shall be equal to the whole $EDCBA$: and the angle FED stands upon the circumference

Book IV. circumference $FABCD$, and the angle AFE upon $EDCBA$ therefore the angle AFE is equal to FED : in the same manner it may be demonstrated that the other angles of the hexagon $ABCDEF$ are each of them equal to the angle AFE or FED ; therefore the hexagon is equiangular; and it is equilateral, as was shown; and it is inscribed in the given circle $ABCDEF$. Which was to be done.

COR. From this it is manifest, that the side of the hexagon is equal to the straight line from the centre, that is, to the radius or semidiameter of the circle.

And if through the points A, B, C, D, E, F there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

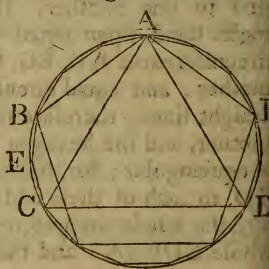
PROP. XVI. PROB.

TO inscribe an equilateral and equiangular quindecagon in a given circle.

Let $ABCD$ be the given circle; it is required to inscribe an equilateral and equiangular quindecagon in the circle $ABCD$.

a 2. 4. Let AC be the side of an equilateral triangle inscribed in the circle, and AB the side of an equilateral and equiangular pentagon inscribed in the same; therefore, of such equal parts as the whole circumference $ABCDEF$ contains fifteen, the circumference ABC , being the third part of the whole, contains five; and the circumference AB , which is the fifth part of the whole, contains three; therefore

c 30. 3. BC their difference contains two of the same parts: bisect



C in E; therefore BE, EC are, each of them, the fifteenth part of the whole circumference ABCD: therefore if the straight lines BE, EC be drawn, and straight lines equal to them be placed ^d around in the whole circle, an equilateral and equiangular quindecagon shall be inscribed in it. Which was to be done.

Book IV.

d. I. 4.

And in the same manner as was done in the pentagon, if through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it: and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

E L E-

The first of these is the fact that the
 of the world is not a uniform one
 but is divided into many different
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 but is divided into many different
 parts.

E L E M E N T S

O F

G E O M E T R Y.

B O O K V.

D E F I N I T I O N S.

I.

A Less magnitude is said to be a part of a greater magnitude, when the less measures the greater, that is, when the less is contained a certain number of times exactly in the greater. Book V.

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

III.

A ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity.

IV.

Magnitudes are said to be of the same kind, when the less can be multiplied so as to exceed the greater; and it is only such magnitudes that are said to have a ratio to one another.

K

V.

Book V.

V.

See N. If there be four magnitudes, and if any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever of the second and fourth, and if, according to the multiple of the first is greater than the multiple of the second, equal to it, or less, the multiple of the third is also greater than the multiple of the fourth, equal to it, or less, then the first of the magnitudes is said to have to the second the same ratio that the third has to the fourth.

VI.

Magnitudes are said to be proportionals, when the first has the same ratio to the second that the third has to the fourth, and the third to the fourth the same ratio which the first has to the second, and so on, whatever be their number.

When four magnitudes, A, B, C, D are proportionals, it is usual to say that A is to B as C to D, and to write them thus, $A : B :: C : D$, or thus, $A : B = C : D$.

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

VIII.

When there is any number of magnitudes greater than two of which the first has to the second the same ratio that the second has to the third, and the second to the third the same ratio which the third has to the fourth, and so on, the magnitudes are said to be continual proportionals.

IX.

When three magnitudes are continual proportionals, the second is said to be a mean proportional between the other two.

X.

When there is any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if $A : B :: E : F$; and $B : C :: G : H$, and $C : D :: K : L$, then, since by this definition A has to D the ratio compounded of the ratios of A to B, B to C, C to D; A may also be said to have to D the ratio compounded of the ratios which are the same with the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D, then, for shortness sake, M is said to have to N a ratio compounded of the same ratios which compound the ratio of A to D; that is, a ratio compounded of the ratios of E to F, G to H, and K to L.

XI.

A ratio which is compounded of two equal ratios is said to be duplicate of either of these ratios.

PROPOSITION. Hence, if the three magnitudes A, B, and C are continual proportionals, the ratio of A to C is duplicate of that of A to B, or of B to C: For by the last definition, the ratio of A to C is compounded of the ratios of A to B, and of B to C; but the ratio of A to B is equal to the ratio of B to C, because A, B, C are continual proportionals; therefore the ratio of A to C, by this definition, is duplicate of the ratio of A to B, or of B to C.

XII.

A ratio which is compounded of three equal ratios is said to be triplicate of any one of these ratios; and a ratio which is compounded of four equal ratios is said to be quadruplicate of any one of these ratios; and so on, according to the number of equal ratios.

COR. If four magnitudes, A, B, C, D be continual proportionals, the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D. For the ratio of A to D is compounded of the three ratios of A to B, B to C, C to D, and these three ratios are equal to one another, because A, B, C, D are continual proportionals, therefore the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D.

XIII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

Geometers make use of the following technical words to signify certain ways of changing either the order or magnitude of proportionals, so as that they continue still to be proportionals.

XIV.

Permutando, or alternando, by permutation, or alternately this word is used when there are four proportionals, and it is inferred, that the first has the same ratio to the third, which the second has to the fourth; or that the first is to the third, as the second to the fourth: As is shewn in the 16th prop. of this 5th book.

XV.

Invertendo, by inversion: When there are four proportionals, and it is inferred, that the second is to the first, as the fourth to the third. Prop. A. book 5.

XVI.

Componendo, by composition: When there are four proportionals, and it is inferred, that the first, together with the second, is to the third, as the fourth, together with the fifth, is to the sixth.

$$\begin{array}{l} \text{XVI} \left\{ \begin{array}{l} 4 A:B::C:D \\ A+B:B::C+D:D \end{array} \right. \end{array}$$

second, is to the second, as the third, together with the Book V.
fourth, is to the fourth. 18th prop. book 5.

XVII.

Dividendo, by division: When there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. 17th prop. book 5.

$$A:B::C:D$$

$$A-B:B::C-D$$

XVIII.

Convertendo, by conversion: When there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Prop. D. book 5.

$$A:A-B::C:C-D$$

XIX.

Ex aequali (sc. distantia), or ex aequo, from equality of distance; when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken two and two.

XX.

Ex aequali, from equality; this term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order, and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in the 22d prop. book 5.

XXI.

Ex aequali, in proportione perturbata, seu inordinata; from equality, in perturbate or disorderly proportion; this term is used when the first magnitude is to the second of

Book V.

the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, is the last but two to the last but one of the second rank, and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank, and so on in a cross order; and the inference is as in the 19th definition. It is demonstrated in the 23d prop. book 5.

IN the demonstrations of this book there are certain *signs* or *characters* which it has been found convenient to employ.

1. The letters A, B, C, &c. are used to denote magnitude of any kind. The letters m , n , p , q , are used to denote numbers only.

2. The sign $+$ (plus), written between two letters that denote magnitudes or numbers, signifies the sum of those magnitudes or numbers. Thus, $A + B$ is the sum of the two magnitudes denoted by the letters A and B; $m + n$ is the sum of the numbers denoted by m and n .

3. The sign $-$ (minus), written between two letters signifies the excess of the magnitude denoted by the first of these letters, which is supposed the greatest, above that which is denoted by the other. Thus, $A - B$ signifies the excess of the magnitude A above the magnitude B.

4. When a number, or a letter denoting a number, is written close to another letter denoting a magnitude of any kind, it signifies that the magnitude is multiplied by the number. Thus, $3A$ signifies three times A; mB m times B, or the multiple of B by m . When the number is intended to multiply two or more magnitudes that follow, it is written thus $m.A + B$, which signifies the sum of A and B taken m times. $m.A - B$ is m times the excess of A above B.

Also, when two letters that denote numbers are written close to one another, they denote the product of those numbers, when multiplied into one another. Thus, mn is the product of m into n .

5. The sign $=$ signifies the equality of the magnitudes denoted by the letters that stand on the opposite sides of it; $A=B$ signifies that A is equal to B ; $A+B=C-D$ signifies that the sum of A and B is equal to the excess of C above D . Book V.

6. The sign $>$ is used to signify the inequality of the magnitudes between which it is placed, and that the magnitude to which the opening of the lines is turned is greater than the other. Thus, $A > B$ signifies that A is greater than B ; $A < B$ signifies that A is less than B . This sign is used, too, to express the inequality of ratios, thus, $A : B > C : D$ signifies that the ratio of A to B is greater than that of C to D ; and $A : B < C : D$ signifies that the ratio of A to B is less than that of C to D .

A X I O M S.

I.

EQUIMULTIPLES of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

IF any number of magnitudes be equimultiples of as many others, each of each, what multiple soever any one of the first is of its part, the same multiple is the sum of all the first of the sum of all the rest.

Let any number of magnitudes A , B , and C be equimultiples of as many others D , E , and F , each of each; $A+B+C$ is the same multiple of $D+E+F$, that A is of D .

Let A contain D , B contain E , and C contain F , each any number of times, as, for instance, three times. Then, because A contains D three times $A = D+D+D$.

For the same reason, $B = E+E+E$;

And also, $C = F+F+F$.

^a Ax. 2. 1. Therefore, adding equals to equals, $A+B+C$ is equal to $D+E+F$ taken three times. In the same manner, if A , B , and C were each any other equimultiple of D , E , and F , it would be shewn that $A+B+C$ was the same multiple of $D+E+F$. Therefore, &c. Q. E. D.

COR. Hence, if m be any number, $mD+mE+mF = m.D+E+F$. For mD , mE , and mF are multiples of D , E , and F by m , therefore their sum is also a multiple of $D+E+F$ by m .

PROP. II. THEOR.

IF to a multiple of a magnitude by any number a multiple of the same magnitude by any number be added, the sum will be the same multiple of that magnitude that the sum of the two numbers is of unity.

Let $A = mC$, and $B = nC$; $A+B = m+n.C$.

For, since $A = mC$, $A = C+C+C+ \&c.$ C being repeated m times. For the same reason, $B = C+C+ \&c.$ C being repeated

peated n times. Therefore, adding equals to equals, $A+B$ is equal to C taken $m+n$ times; that is, $A+B = m+n.C$. Therefore $A+B$ contains C as oft as there are units in $m+n$. Q. E. D.

COR. 1. In the same way, if there be any number of multiples whatsoever, as $A = mE$, $B = nE$, $C = pE$, it is shewn, that $A+B+C = m+n+p.E$.

COR. 2. Hence also, since $A+B+C = m+n+p.E$, and since $A = mE$, $B = nE$, and $C = pE$, $mE+nE+pE = m+n+p.E$.

PROP. III. THEOR.

IF the first of three magnitudes contain the second as oft as there are units in a certain number, and the second contain the third also as often as there are units in a certain number, the first will contain the third as oft as there are units in the product of these two numbers.

Let $A = mB$, and $B = nC$; then $A = mnC$.

Since $B = nC$, $mB = nC+nC+\&c.$ repeated m times. But $C+nC+\&c.$ repeated m times, is equal ^a to C multiplied by $m+n+\&c.$ n being repeated m times. And n repeated m times is mn ; therefore $mB = mnC$. But by hypothesis, $A = mB$, therefore $A = mnC$.

PROP. IV. THEOR.

IF the first of four magnitudes has the same ratio to the second which the third has to the fourth, and if any equimultiples whatever be taken of the first and third, and any whatever of the second and fourth; the multiple of the first shall have the same ratio to the multiple of the second that the multiple of the third has to the multiple of the fourth.

Let

Book V.

Let $A : B :: C : D$, and let m and n be any two numbers ;
 $mA : nB :: mC : nD$.

Take of mA and mC equimultiples by any number p , and of nB and nD equimultiples by any number q . Then the equimultiples of mA and mC by p , are equimultiples also of A and C , for they contain A and C as oft as there are units in pm^a , and are equal to pmA and pmC . For the same reason the multiples of nB and nD by q , are qnB , qnD . Since therefore $A : B :: C : D$, and of A and C there are taken any equimultiples, viz. pmA and pmC , and of B and D , any equimultiples qnB , qnD , if pmA be greater than qnB , pmC must be greater than qnD ^b; if equal, equal; and if less, less. But pmA , pmC are also equimultiples of mA and mC , and qnB , qnD are equimultiples of nB and nD , therefore^b, $mA : nB :: mC : nD$. Therefore, &c. Q. E. D.

COR. In the same manner it may be demonstrated, that if $A : B :: C : D$, and of A and C equimultiples be taken by any number m , viz. mA and mC , $mA : B :: mC : D$. This may also be considered as included in the proposition, and as being the case when $n = 1$.

PROP. V. THEOR.

IF one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Let mA and mB be any equimultiples of the two magnitudes A and B , of which A is greater than B ; $mA - mB$ is the same multiple of $A - B$ that mA is of A , that is, $mA - mB = m.A - B$.

Let D be the excess of A above B , then $A - B = D$, and adding B to both $A = D + B$. Therefore^a $mA = mD + mB$; take mB from both, and $mA - mB = mD$; but $D = A - B$, therefore $mA - mB = m.A - B$. Therefore, &c. Q. E. D.

PROP.

PROP. VI. THEOR.

IF from a multiple of a magnitude by any number a multiple of the same magnitude by a less number be taken away, the remainder will be the same multiple of that magnitude that the difference of the numbers is of unity.

Let mA and nA be multiples of the magnitude A by the numbers m and n , and let m be greater than n ; $mA - nA$ contains A as oft as $m - n$ contains unity, or $mA - nA = \overline{m - n}.A$.

Let $m - n = q$; then $m = n + q$. Therefore ^a $mA = nA + qA$; ^{a 2. 5.} take nA from both, and $mA - nA = qA$. Therefore $mA - nA$ contains A as oft as there are units in q , that is, in $m - n$, or $mA - nA = \overline{m - n}.A$. Therefore, &c. Q. E. D.

COR. When the difference of the two numbers is equal to unity, or $m - n = 1$, then $mA - nA = A$.

PROP. A. THEOR.

IF four magnitudes be proportionals, they are proportionals also when taken inversely.

If $A : B :: C : D$, then also $B : A :: D : C$.

Let mA and mC be any equimultiples of A and C ; nB and nD any equimultiples of B and D . Then, because $A : B :: C : D$, if mA be less than nB , mC will be less than nD ^{a, a def. 5. 5.} that is, if nB be greater than mA , nD will be greater than mC . For the same reason, if $nB = mA$, $nD = mC$, and if $nB < mA$, $nD < mC$. But nB , nD are any equimultiples of B and D , and mA , mC any equimultiples of A and C , therefore ^a, $B : A :: D : C$. Therefore, &c. Q. E. D.

PROP.

PROP. B. THEOR.

IF the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second as the third to the fourth.

First, if mA , mB be equimultiples of the magnitudes A and B , $mA : A :: mB : B$.

Take of mA and mB equimultiples by any number n ; and of A and B equimultiples by any number p ; these will be nmA^a , pA , nmB^a , pB . Now, if nmA be greater than pA , nm is also greater than p ; and if nm is greater than p , nmB is greater than pB ; therefore, when nmA is greater than pA , nmB is greater than pB . In the same manner, if $nmA = pA$, $nmB = pB$, and if $nmA < pA$, $nmB < pB$. Now, nmA , nmB are any equimultiples of mA and mB ; and pA , pB are any equimultiples of A and B , therefore $mA : A :: mB : B$.
 Next, Let C be the same part of A that D is of B ; then A is the same multiple of C that B is of D ; and therefore as has been demonstrated, $A : C :: B : D$, and inversely $C : A :: D : B$. Therefore, &c. Q. E. D.

PROP. VI. THEOR.

IF the first be to the second as the third to the fourth; and if the first be a multiple or a part of the second, the third is the same multiple or the same part of the fourth.

Let $A : B :: C : D$, and first let A be a multiple of B , C is the same multiple of D ; that is, if $A = mB$, $C = mD$.

Take of A and C equimultiples by any number as 2, viz. $2A$ and $2C$; and of B and D , take equimultiples by the number

number $2m$, viz. $2mB$, $2mD$ ^a; then, because $A = mB$, $2A = 2mB$; and since $A : B :: C : D$, and since $2A = 2mB$, therefore $2C = 2mD$ ^b, and $C = mD$, that is, C contains D m times, or as often as A contains B . Book V.
a 3. 5.
b def. 5. 5.

Next, Let A be a part of B , C is the same part of D . For, since $A : B :: C : D$, inversely ^c, $B : A :: D : C$. But A being a part of B , B is a multiple of A , and therefore, as is shewn above, D is the same multiple of C , and therefore C is the same part of D that A is of B . Therefore, &c. Q. E. D. c A. 5.

PROP. VII. THEOR.

EQUAL magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other; $A : C :: B : C$.

Let mA , mB be any equimultiples of A and B ; and nC any multiple of C .

Because $A = B$, $mA = mB$ ^a; wherefore, if mA be greater than nC , mB is greater than nC ^b; and if $mA = nC$, $mB = nC$; or, if $mA < nC$, $mB < nC$. But mA and mB are any equimultiples of A and B , and nC is any multiple of C , therefore ^b $A : C :: B : C$. a A. 1. 5.
b def. 5. 5.

Again, if $A = B$, $C : A :: C : B$; for, as has been proved, $A : C :: B : C$, and inversely ^c, $C : A :: C : B$. Therefore, &c. Q. E. D. c A. 5.

PROP. VIII. PROB.

OF unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Let

I

Book V.

Let $A+B$ be a magnitude greater than A and C a third magnitude, $A+B$ has to C a greater ratio than A has to C and C has a greater ratio to A than it has to $A+B$.

Let m be such a number that mA and mB are each of them greater than C ; and let nC be the least multiple of C that exceeds $mA+mB$; then $nC-C$, that is, $n-1$. C will be less than $mA+mB$, or $mA+nB$, that is m . $A+B$ is greater than $n-1$. C . But because nC is greater than $mA+mB$, and C less than mB , $nC-C$ is greater than mA , or n . A is less than $nC-C$, that is, than $n-1$. C . Therefore the multiple of $A+B$ by m exceeds the multiple of C by $n-1$, but the multiple of A by m does not exceed the multiple of C by $n-1$.
 a 1. 5. therefore $A+B$ has a greater ratio to C than A has to C .
 b 7. def. 5.

Again, because the multiple of C by $n-1$, exceeds the multiple of A by m , but does not exceed the multiple of $A+B$ by m , C has a greater ratio to A than it has to $A+B$. Therefore, &c. Q. E. D.

PROP. IX. THEOR.

MAGNITUDES which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

If $A : C :: B : C$, $A = B$.

For if not, let A be greater than B ; then, because A is greater than B , two numbers, m and n , may be found as in the last proposition, such that mA shall exceed nC , while mB does not exceed nC . But because $A : C :: B : C$; if mA exceeds nC , mB must also exceed nC ; and it is also shewn that mB does not exceed nC , which is impossible. Therefore A is not greater than B ; and in the same way it is demonstrated that B is not greater than A ; therefore A is equal to B .
 a 5. def. 5.

Next, let $C : A :: C : B$, $A = B$. For by inversion $A : C :: B : C$; and therefore by the first case, $A = B$.
 b A. 5.

PROP.

PROP. X. THEOR.

THAT magnitude which has a greater ratio than another has unto the same magnitude is the greater of the two: And that magnitude to which the same has a greater ratio than it has unto another magnitude is the lesser of the two.

If the ratio of A to C be greater than that of B to C , A is greater than B .

Because $A : C > B : C$, two numbers m and n may be found, such that $mA > nC$, and $mB < nC$ ^a. Therefore also $mA > mB$, and $A > B$ ^b. a def. 7. 5.
b 4. Ax. 5.

Again, let $C : B > C : A$; $B < A$. For two numbers, m and n may be found, such that $mC > mB$, and $mC < mA$ ^a. Therefore, since mB is less, and mC greater than the same magnitude, $mB < mA$, and therefore $B < A$.

PROP. XI. THEOR.

RATIOS that are equal to the same ratio are equal to one another.

If $A : B :: C : D$; and also $C : D :: E : F$; then $A : B :: E : F$.

Take mA , mC , mE any equimultiples of A , C , and E ; and nB , nD , nF any equimultiples of B , D , and F . Because $A : B :: C : D$, if $mA > nB$, $mC > nD$ ^a; but if $mC > nD$, $mE > nF$ ^a, because $C : D :: E : F$; therefore if $mA > nB$, $mE > nF$. In the same manner, if $mA = nB$, $mE = nF$; and if $mA < nB$, $mE < nF$. Now, mA , mE are any equimultiples whatever of A and E ; and nB , nF any whatever of B and F ; therefore $A : B :: E : F$ ^a. Therefore, &c. Q. E. D.

PROP.

can be

PROP. XII. THEOR.

IF any number of magnitudes be proportionals, and one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

If $A : B :: C : D$, and $C : D :: E : F$; $A : B :: A + C + E : B + D + F$.

Take mA , mC , mE any equimultiples of A , C , and E and nB , nD , nF , any equimultiples of B , D , and F . Then, because $A : B :: C : D$, if $mA > nB$, $mC > nD$ ^a; and when $mC > nD$, $mE > nF$, because $C : D :: E : F$. Therefore, if $mA > nB$, $mA + mC + mE > nB + nD + nF$: In the same manner, if $mA = nB$, $mA + mC + mE = nB + nD + nF$; and if $mA < nB$, $mA + mC + mE < nB + nD + nF$. Now $mA + mC + mE$ is m times $A + C + E$ ^b, so that mA and $mA + mC + mE$ are any equimultiples of A and of $A + C + E$. And for the same reason nB , and $nB + nD + nF$ are any equimultiples of B , and of $B + D + F$; therefore^a $A : B :: A + C + E : B + D + F$. Therefore, &c. Q. E. D.

PROP. XIII. PROB.

IF the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth, the first shall also have to the second a greater ratio than the fifth has to the sixth.

If $A : B :: C : D$; but $C : D > E : F$; then also, $A : B > E : F$.

Because $C : D > E : F$, there are some two numbers m and n such that $mC > nD$, but $mE < nF$ ^a. Now, if $mC > nD$, $mA > nB$, because $A : B :: C : D$. Therefore $mA > nB$, and $mE < nF$, wherefore, $A : B > E : F$. Therefore, &c. Q. E. D.

PROB.

PROP. XIV. THEOR.

IF the first has to the second the same ratio which the third has to the fourth, then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

If $A : B :: C : D$; then if $A > C$, $B > D$, if $A = C$, $B = D$, and if $A < C$, $B < D$.

First, let $A > C$; then $A : B > C : B^a$, but $A : B :: C : D$, therefore $C : D > C : B^b$, and therefore $B > D^c$.

In the same manner, it is proved, that if $A = C$, $B = D$; and if $A < C$, $B < D$. Therefore, &c. Q. E. D.

^a 8. 5.
^b 13. 5.
^c 10. 5.

PROP. XV. THEOR.

MAGNITUDES have the same ratio to one another which their equimultiples have.

If A and B be two magnitudes, and m any number, $A : B :: mA : mB$.

Because $A : B :: A : B^a$; $A : B :: A + A : B + B^b$, or $A : B :: 2A : 2B$. And since it has been proved that $A : B :: A : 2B$, $A : B :: A + 2A : B + 2B^b$, or $A : B :: 3A : 3B$; and so on, for all the equimultiples of A and B . Therefore, &c. Q. E. D.

^a 7. 5.

^b 12.

PROP. XVI. THEOR.

IF four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

L

If

Book V.

If $A : B :: C : D$, then alternately, $A : C :: B : D$.

Take mA , mB any equimultiples of A and B , and nC , nD any equimultiples of C and D . Then^a $A : B :: mA : mB$ now $A : B :: C : D$, therefore^b $C : D :: mA : mB$. But $C : D :: nC : nD$ ^a; therefore $mA : mB :: nC : nD$ ^b; wherefore if $mA > nC$, $mB > nD$ ^c; if $mA = nC$, $mB = nD$, or if $mA < nC$, $mB < nD$; therefore, $A : C :: B : D$. Therefore, &c. Q. E. D.

PROP. XVII. THEOR.

IF magnitudes, taken jointly, be proportionals they shall also be proportionals when taken separately; that is, if the first, together with the second, have to the second the same ratio which the third, together with the fourth, has to the fourth the first shall have to the second the same ratio which the third has to the fourth.

If $A+B : B :: C+D : D$, then by division $A : B :: C : D$.Take mA and nB any equimultiples of A and B , by the numbers m and n ; and first, let $mA > nB$: to each of these^a 1. 5. add mB , then $mA+mB > mB+nB$. But $mA+mB = m.(A+B)$ ^b 2. 5. and $mB+nB = m+n.B$ ^b, therefore $m.(A+B) > m+n.B$.

And because $A+B : B :: C+D : D$, if $m.(A+B) > m+n.B$ then $m.(C+D) > m+n.D$, or $mC+mD > mD+nD$, that is, taking mD from both, $mC > nD$. Therefore, when mA is greater than nB , mC is greater than nD . In like manner it is demonstrated, that if $mA = nB$, $mC = nD$, and if $mA < nB$, $mC < nD$; therefore $A : B :: C : D$. Therefore, &c. Q. E. D.

PROP

PROP. XVIII. THEOR.

If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly, that is, if the first be to the second as the third to the fourth, the first and second together shall be to the second, as the third and fourth together to the fourth.

If $A:B::C:D$, then, by composition, $A+B:B::C+D:D$.

Take $m.A+B$, and nB any multiples whatever of $A+B$ and B : and first, let m be greater than n . Then, because $A+B$ is also greater than B , $m.A+B > n.B$. For the same reason, $m.C+D > nD$. In this case, therefore, that is, when $m > n$, $m.A+B$ is greater than nB , and $m.C+D$ is greater than nD . And in the same manner it may be proved, that when $m = n$, $m.A+B$ is greater than nB , and $m.C+D$ greater than nD .

Next, let $m < n$, or $n > m$, then $m.A+B$ may be greater than nB , or may be equal to it, or may be less; first, let $A+B$ be greater than nB ; then also, $m.A+B > nB$; take mB which is less than nB , from both, and $m.A > nB - mB$, $m.A > n - m.B^a$. But if $m.A > n - m.B$, $m.C > n - m.D$; a 6. 5. because $A:B::C:D$. Now, $n - m.D = nD - mD$ a, therefore, $m.C > nD - mD$, and adding mD to both $m.C + mD > nD$, that is b $m.C+D > nD$. If therefore, $m.A+B$ b 1. 5. $m.C+D > nD$.

In the same manner it will be proved, that if $m.A+B = nB$, $m.C+D = nD$; and if $m.A+B < nB$, $m.C+D < nD$; therefore, $A+B:B::C+D:D$. Therefore, &c. Q. E. D.

PROP. XIX. THEOR.

IF a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

If $A : B :: C : D$, and if C be less than A , $A - C : B - C :: A : B$.

^a 16. 5. Because $A : B :: C : D$, alternately ^a, $A : C :: B : D$; and
^b 17. 5. therefore by division ^b $A - C : C :: B - D : D$. Wherefore again, alternately, $A - C : B - D :: C : D$; but $A : B :: C : D$; therefore $A - C : B - D :: A : B$. Therefore, &c. Q. E. D.
 Cor. $A - C : B - D :: C : D$.

PROP. D. THEOR.

IF four magnitudes be proportionals, they are all proportionals by conversion, that is, the first is to its excess above the second, as the third to its excess above the fourth.

If $A : B :: C : D$, by conversion, $A : A - B :: C : C - D$.
^a 17. 5. For since $A : B :: C : D$, by division ^a, $A - B : B :: C - D : D$, and inversely ^b, $B : A - B :: D : C - D$; therefore
^b A. 5. by composition ^c, $A : A - B :: C : C - D$. Therefore, &c.
^c 18. 5. Q. E. D.

PROP.

PROP. XX. THEOR.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

If there be three magnitudes, A, B, and C, and other three D, E, and F; and if $A : B :: D : E$; and also $B : C :: E : F$, then if $A > C$, $D > F$; if $A = C$, $D = F$; and if $A < C$, $D < F$.

A,	B,	C,
D,	E,	F.

First, let $A > C$; then $A : B > C : B$ ^a. But $A : B :: D : E$, therefore also $D : E > C : B$ ^b. Now $B : C :: E : F$, and inferably $C : B :: F : E$; and it has been shewn that $D : E > C : B$; therefore $D : E > F : E$ ^c, and consequently $D > F$ ^d. Next, let $A = C$; then $A : B :: C : B$ ^e, but $A : B :: D : E$, therefore, $C : B :: D : E$; but $C : B :: F : E$, therefore, $D : E :: F : E$ ^f, and $D = F$ ^g. Lastly, let $A < C$. Then $A > A$, and, as was shewn in the first case, $C : B :: F : E$, and $A : B :: E : D$, therefore, by the first case, if $C > A$, $F > D$, that is, if $A < C$, $D < F$. Therefore, &c. Q. E. D.

a 8. 5.
b 13. 5.
c 4. 5.
d 10. 5.
e 7. 5.
f 11. 5.
g 9. 5.

PROP. XXI. THEOR.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

If there be three magnitudes, A, B, C, and other three, D, E, and F, such that $A : B :: E : F$, and $B : C :: D : E$; if $A > C$, $D > F$; if $A = C$, $D = F$, and if $A < C$, $D < F$.

I. 3

First,

Book V.

First, let $A > C$. Then $A : B > :$ $C : B$ ^a, but $A : B :: E : F$, therefore, $E : F > C : B$ ^b. Now, $B : C :: D : E$,and inversely, $C : B :: E : D$; there-fore, $E : F > E : D$ ^b, wherefore, $D > F$ ^c.

d 7. 5.

Next, let $A = C$. Then $A : B :: C : B$; but $A : B :: E : F$, therefore, $C : B :: E : F$ ^d; but $B : C :: D : E$, and

e 11. 5

inversely, $C : B :: E : D$, therefore ^e, $E : F :: E : D$, and, c

f 9. 5.

sequently, $D = F$ ^f.

Lastly, let $A < C$. Then $C > A$, and, as was proved in first case, $C : B :: E : D$; and $B : A :: F : E$, therefore, the first case, since $C > A$, $F > D$, that is, $D < F$. Therefore, &c. Q. E. D.

A,	B,	C,
D,	E,	F.

PROP. XXII. THEOR.

IF there be any number of magnitudes, and many others, which, taken two and two in order have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last*.

First, let there be three magnitudes, A, B, C, and three, D, E, F, which, taken two and two in order, have the same ratio, viz. $A : B :: D : E$, and $B : C :: E : F$; therefore, $A : C :: D : F$.

Take of A and D any equimultiples whatever, mA , mD ; of B and E any whatever, nB , nE ; and of C and F any whatever, qC , qF . Because $A : B :: D : E$, $mA : nB :: mD : nE$ ^a; and for

a 4. 5.

the same reason, $nB : qC :: nE : qF$.

b 20. 5

Therefore ^b, according as mA is greater than qC , equal to it, or less, mB isgreater than qF , equal to it, or less; but mA , mD are any equimultiples of A and D; and

c def. 5. 5.

 qF are any equimultiples of C and F; therefore ^c, $A : C :: D : F$.

A,	B,	C,
D,	E,	F,
mA ,	nB ,	qC ,
mD ,	nE ,	qF .

Agg

* N. B. This proposition is usually cited by the words "ex æquali," "ex æquo."

Book V.

Again, let there be four magnitudes, and other four, which, two and two, have the same ratio, viz. $A : B :: E : F$; $B : C :: F : G$; $C : D :: G : H$, then $A : D :: E : H$.

For since A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio, by the foregoing case $A : C :: E : G$. And because also $C : D :: G : H$, by that same case, $A : D :: E : H$. In the same manner is the demonstration extended to any number of magnitudes. Therefore, &c. Q. E. D.

$A, B, C, D,$
$E, F, G, H.$

PROP. XXIII. THEOR.

IF there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last*.

First, let there be three magnitudes, A, B, C , and other three, D, E , and F , which, taken two and two in a cross order, have the same ratio, viz. $A : B :: E : F$, and $B : C :: D : E$, then $A : C :: D : F$.

Take of A, B , and D any equimultiples mA, mB, mD ; and of C, E, F any equimultiples nC, nE, nF .

Because $A : B :: E : F$, and because also $A : B :: mA : mB$ ^a, ^{a 15. 5.} and $E : F :: nE : nF$ therefore, $mA : mB :: nE : nF$ ^b. ^{b 11. 5.} A-

gain, because $B : C :: D : E$, $mB : nC :: mD : nE$; and it has been just shewn that $mA : mB :: nE : nF$; therefore, if $mA > nC$, $mD > nF$ ^c; if $mA = nC$, $mD = nF$; and if $mA < nC$, $mD < nF$. Now, mA and mD are any equimultiples of A and D , and nC, nF any equimultiples of C and F ; therefore, $A : C :: D : F$ ^d.

$A, B, C,$
$D, E, F,$
$mA, mB, nC,$
$mD, nE, nF.$

c 21. 5.

d 5. def 5.

L 4

Next,

* N. B. This proposition is usually cited by the words, "ex æquali in proportione perturbata;" or, "ex æquo perturbatè."

Book V.

Next, let there be four magnitudes A, B, C, and D, and other four, E, F, G, and H, which, taken two and two in cross order, have the same ratio, viz.

$A : B :: G : H$; $B : C :: F : G$, and

$C : D :: E : F$, then $A : D :: E : F$.

For since A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, in a cross order, have the same ratio, by the first case $A : C :: E : G$. But $C : D :: G : H$, therefore again, by the first case $A : D :: E : H$. In the same manner may the demonstration be extended to any number of magnitudes. Therefore, &c. Q. E. D.

A,	B,	C,	D,
E,	F,	G,	H.

PROP. XXIV. THEOR.

IF the first has to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let $A : B :: C : D$, and also $E : B :: F : D$, then $A + E : B :: C + F : D$.

Because $B : E :: D : F$, by inversion, $E : B :: F : D$.

a 22. 5. But by hypothesis, $A : B :: C : D$, therefore, ex æquali

b 18. 5. $A : E :: C : F$, and by composition^b, $A + E : E :: C + F : F$.

Now again by hypothesis, $B : E :: D : F$, therefore, ex æquali^a $A + E : B :: C + F : D$. Therefore, &c. Q. E. D.

PROP. E. THEOR.

RATIOS which are compounded of the same ratios, are the same with one another.

Let

Let the ratios of A to B, and of B to C, which compound Book V.
the ratio of A to C, be equal, each to each, to the ratios of D ~
to E, and E to F, which compound the ratio of E to F,
 $A : C :: E : F$.

For, first, if the ratio of A to B be
equal to that of D to E, and the ratio
of B to C equal to that of E to F, ex
equali^a, $A : C :: D : F$.

A,	B,	C,
D,	E,	F.

^a 22. 5.

And next, if the ratio of A to B be equal to that of E to F,
and the ratio of B to C equal to that of D to E, ex æquali in
proportionē perturbata^b, $A : C :: D : F$. In the same man- b 23. 5.
ner may the proposition be demonstrated, whatever be the
number of ratios. Therefore, &c. Q. E. D.

E L E

E L E M E N T S

O F

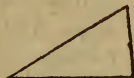
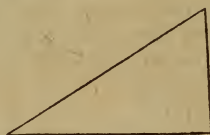
G E O M E T R Y.

B O O K VI.

D E F I N I T I O N S.

I.

SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.



Book VI.

II.

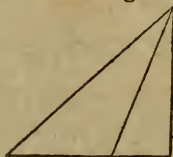
Two sides of one figure are said to be reciprocally proportional to two sides of another, when one of the sides of the first is to one of the sides of the other, as the remaining side of the other is to the remaining side of the first.

III.

A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

IV.

The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.



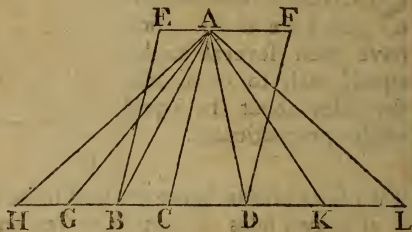
P R O P.

PROP. I. THEOR.

TRIANGLES and parallelograms of the same altitude are one to another as their bases.

Let the triangles ABC , ACD , and the parallelograms EC , CF have the same altitude, viz. the perpendicular drawn from the point A to BD : Then, as the base BC is to the base CD so is the triangle ABC to the triangle ACD , and the parallelogram EC to the parallelogram CF .

Produce BD both ways to the points H , L , and take any number of straight lines BG , GH , each equal to the base BC and DK , KL , any number of them, each equal to the base CD ; and join AG , AH , AK , AL : Then, because CB , BG , GH are all equal, the triangles AHG , AGB , ABC are all equal ^a: Therefore, whatever multiple the base HC is of the base BC , the same multiple is the triangle AHC of the triangle ABC . For the same reason, whatever multiple the base LC is of the base CD , the same multiple is the triangle ALC of the triangle ADC : And if the base HC be equal to the base CL , the triangle AHC is also equal to the triangle ALC ^a; and if the base HC be greater than the base CL , likewise the triangle AHC is greater than the triangle ALC ; and if less, less. Therefore, since there are four magnitudes, viz. the two bases BC , CD , and the two triangles ABC , ACD ; and of the base BC and the triangle ABC , the first and third, any equimultiples whatever have been taken, viz. the base HC and triangle AHC ; and of the base CD and triangle ACD , the second and fourth, have been taken any equimultiples whatever, viz. the base CL and triangle ALC ; and since it has been shown, that if the base HC be greater than the base CL , the triangle AHC is greater than the triangle



angle ALC; and if equal, equal; and if less, less: There- Book VI.
fore ^b, as the base BC is to the base CD, so is the triangle b 5. def. 5.
ABC to the triangle ACD.

And because the parallelogram CE is double of the triangle
ABC ^c; and the parallelogram CF double of the triangle c 41. 1.
ACD, and that magnitudes have the same ratio which their
equimultiples have ^d; as the triangle ABC is to the triangle d 15. 5.
ACD, so is the parallelogram EC to the parallelogram CF.
And because it has been shown, that, as the base BC is to the
base CD, so is the triangle ABC to the triangle ACD; and
as the triangle ABC to the triangle ACD, so is the parallelo-
gram EC to the parallelogram CF; therefore, as the base BC
is to the base CD, so is ^e the parallelogram EC to the paral- e 11. 5.
lelogram CF. Wherefore triangles &c. Q. E. D.

COR. From this it is plain, that triangles and parallelo-
grams that have equal altitudes, are one to another as their
bases.

Let the figures be placed so as to have their bases in the
same straight line; and having drawn perpendiculars from the
vertices of the triangles to the bases, the straight line which
joins the vertices is parallel to that in which their bases are ^f, f 33. 1.
because the perpendiculars are both equal and parallel to one
another. Then, if the same construction be made as in the
proposition, the demonstration will be the same.

PROP. II. THEOR.

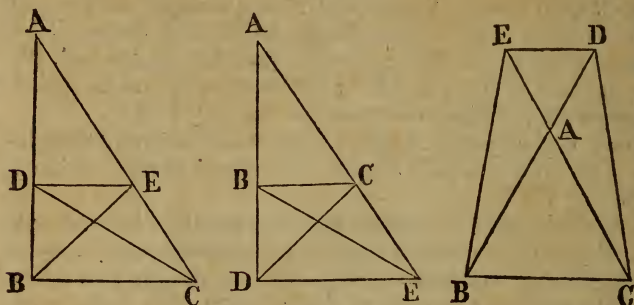
IF a straight line be drawn parallel to one of the
sides of a triangle, it shall cut the other sides, or
the other sides produced, proportionally: And if the
sides, or the sides produced, be cut proportionally,
the straight line which joins the points of section shall
be parallel to the remaining side of the triangle.

Let DE be drawn parallel to BC, one of the sides of the
triangle ABC: BD is to DA, as CE to EA.

Join BE, CD; then the triangle BDE is equal to the triangle
CDE ^a, because they are on the same base DE, and between a 37. 1.
the same parallels DE, BC: but ADE is another triangle,
and

- Book VI.** and equal magnitudes have to the same the same ratio ^b; therefore, as the triangle BDE to the triangle ADE, so is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, so is ^c BD to DA, because having the same altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bases; and for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA. Therefore, as BD to DA, so is CE to EA ^d.

Next, let the sides AB, AC of the triangle ABC, or these



sides produced, be cut proportionally in the points D, E, that is, so that BD be to DA, as CE to EA, and join DE; DE is parallel to BC.

- The same construction being made, because as BD to DA, so is CE to EA; and as BD to DA, so is the triangle BDE to the triangle ADE ^e; and as CE to EA, so is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the same ratio to the triangle ADE; and therefore ^f the triangle BDE is equal to the triangle CDE: And they are on the same base DE; but equal triangles on the same base are between the same parallels ^g; therefore DE is parallel to BC. Wherefore, if a straight line, &c. Q. E. D.

PROP.

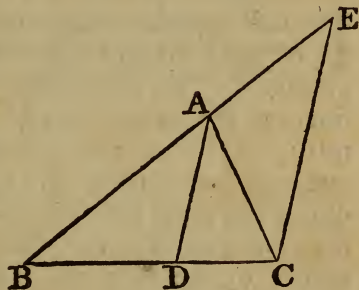
PROP. III. THEOR.

If the angle of a triangle be bisected by a straight line which also cuts the base; the segments of the base shall have the same ratio which the other sides of the triangle have to one another: And if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section, bisects the vertical angle.

Let the angle BAC of any triangle ABC be divided into two equal angles by the straight line AD : BD is to DC , as BA to AC .

Through the point C draw CE parallel^a to DA , and let ^{a 31. 1.} BA produced meet CE in E . Because the straight line AC meets the parallels AD , EC , the angle ACE is equal to the alternate angle CAD ^b: But CAD , by the hypothesis, is ^{b 29. 1.} equal to the angle BAD ; wherefore BAD is equal to the angle ACE . Again, because the straight line BAE meets the parallels AD , EC , the outward angle AD is equal to the inward and opposite angle EC : But the angle ACE has been proved equal to the angle BAD ; therefore ACE is equal to the angle AEC , and consequently the side AE is equal to the side AC :

And because AD is drawn parallel to one of the sides of the triangle BCE , viz. to EC , BD is to DC , as BA to AE ^d; but AE is equal to AC ; therefore, as BD to DC , so is BA to AC .



^{c 6. 1.}

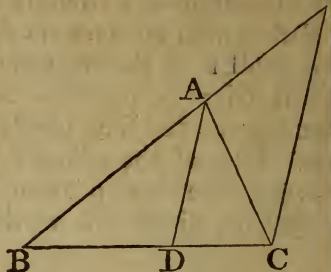
^{d 2. 6.}

^{e 7. 5.}

Let

Book VI. Let now BD be to DC , as BA to AC , and join AD ; the angle BAC is divided into two equal angles by the straight line AD .

The same construction being made; because, as BD to DC so is BA to AC ; and as BD to DC , so is BA to AE^d , because AD is parallel to EC ; therefore AB is to AC , as BA to AE^f : Consequently AC is equal to AE^g , and the angle AEC is therefore equal to the angle ACE^h : But the angle AEC is equal to the outward and opposite angle BAD ; and the angle ACE is equal to the alternate angle CAD^b : Wherefore also the angle BAD is equal to the angle CAD : Therefore the angle BAC is cut into two equal angles by the straight line AD . Therefore, if the angle, &c. Q. E. D.



PROP. A. THEOR.

IF the exterior angle of a triangle be bisected by a straight line which also cuts the base produced; the segments between the bisecting line and the extremities of the base have the same ratio which the other sides of the triangle have to one another: And if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section bisects the exterior angle of the triangle.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the straight line AD which meets the base produced in D : BD is to DC , as BA to AC .

Through

Through C draw CF parallel to AD ^a; and because the straight line AC meets the parallels AD , FC , the angle ACF is equal to the alternate angle CAD ^b: But CAD is equal to the angle DAE ^c; therefore also DAE is equal to the angle ACF . Again, because the straight line FAE meets the parallels AD , FC , the outward angle DAE is equal to the inward and opposite angle CFA : But the angle ACF has been proved to be equal

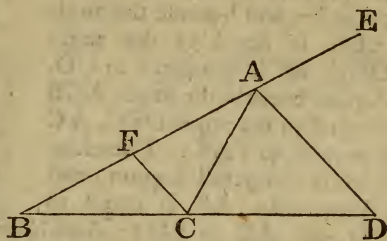
Book VI.

a 31. 1.

b 29. 1.

c Hyp.

the angle DAE ; therefore also the angle CF is equal to the angle CFA , and consequently the side AF is equal to the side AC ^d: and because AD is parallel to FC , a side of the triangle BCF , BD to DC , as BA to AC ^e; but AF is equal to AC ; as therefore BD is to DC , so is BA to AC .



d 6. 1.

is BA to AC .

Let now BD be to DC , as BA to AC , and join AD ; the angle CAD is equal to the angle DAE .

The same construction being made, because BD is to DC , BA to AC ; and also BD to DC , as BA to AF ^e; therefore BA is to AC , as BA to AF ^f; wherefore AC is equal to AF ^g, and the angle AFC equal^h to the angle ACF : but the angle AFC is equal to the outward angle EAD , and the angle ACF to the alternate angle CAD ; therefore also EAD is equal to the angle CAD . Wherefore, if the exterior, &c. Q. E. D.

f 11. 5.

g 9. 5.

h 5. 1.

PROP. IV. THEOR.

THE sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

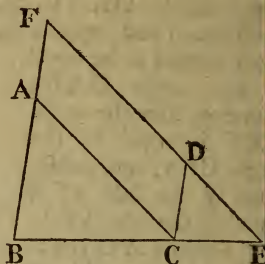
Let ABC , DCE be equiangular triangles, having the angle BCA equal to the angle DCE , and the angle BAC to the angle DEC , and consequently^a the angle BAC equal to the angle CDE . The sides about the equal angles of the triangles ABC , DCE are proportionals; and those are the homologous sides which are opposite to the equal angles.

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Let

Book VI.

- Let the triangle DCE be placed, so that its side CE may be contiguous to BC, and in the same straight line with it. And because the angles ABC, ACB are together less than two right angles, ^b ABC and DEC, which is equal to ACD, are also less than two right angles; wherefore BA, ED produced shall meet ^c; let them be produced and meet in the point F; and because the angle ABC is equal to the angle DCE, BF is parallel ^d to CD. Again, because the angle ACB is equal to the angle DEC, AC is parallel to FE ^d: Therefore FACD is a parallelogram; and consequently AF is equal to CD, and AC to FD ^e: And because AC is parallel to FE, one of the sides of the triangle FBE, $BA : AF :: BC : CE$ ^f; but AF is equal to CD; therefore ^g, $BA : CD :: BC : CE$; and alternately, $AB : BC :: DC : CE$ ^b: Again, because CD is parallel to BF, $BC : CE :: FD : DE$ ^f; but FD is equal to AC; therefore, $BC : CE :: AC : DE$; and alternately, $BC : CA :: CE : ED$. Therefore, because it has been proved that $AB : BC :: DC : CE$; and $BC : CA :: CE : ED$, ex æquali, $BA : AC :: CD : DE$. Therefore the sides, &c. Q. E. D.



PROP. V. THEOR.

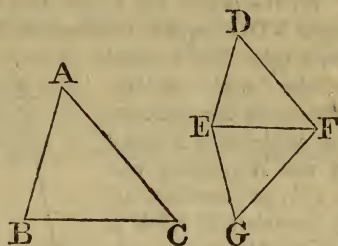
IF the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportionals, so that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and consequently, ex æquali, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides.

z. the angle ABC being equal to the angle DEF , and BCA to FD , and also BAC to EDF . Book VI.

At the points E, F , in the straight line EF , make ^a the angle FEG equal to the angle ABC , and the angle FFG equal to BCA ; wherefore

the remaining angle BAC equal to the remaining angle EGF ^b, and the triangle ABC is therefore equiangular to the triangle EGF ; and consequently they have their sides opposite the equal angles proportionals^c. Wherefore, as AB to BC , so is GE to EF ; and as AB to BC , so is DE



b 32. 1.

c 4. 6.

EF ; therefore as DE to EF , so ^d GE to EF : Therefore DE and GE have the same ratio to EF , and consequently are equal^e: For the same reason, DF is equal to FG : and because, in the triangles DEF, GEF , DE is equal to EG , and EF common, and also the base DF equal to the base GF ; therefore the angle DEF is equal^f to the angle GEF , and the other angles to the other angles, which are subtended by equal sides^g. Wherefore the angle DFE is equal to the angle GFE , and EDF to EGF : and because the angle DEF is equal to the angle GEF , and GEF to the angle ABC ; therefore the angle ABC is equal to the angle DEF : For the same reason, the angle ACB is equal to the angle DFE , and the angle at A to the angle at D . Therefore the triangle ABC is equiangular to the triangle DEF . Wherefore, if the sides, &c. Q. E. D.

d 11. 5.

e 9. 5.

f 8. 1.

g 4. 1.

PROP. VI. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

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Let

Book VI.

Let the triangles ABC , DEF have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals; that is, BA to AC , as ED to DF : the triangles ABC , DEF are equiangular, and have the angle ABC equal to the angle DEF , and ACB to DFE .

a 23. 1. At the points D , F , in the straight line DF , make ^a the angle FDG equal to either of the angles BAC , EDF ; and the angle DFG equal

to the angle ACB ;

wherefore the remain-

ing angle at B is equal

to the remaining one

at G ^b, and consequent-

ly the triangle ABC is

equiangular to the tri-

angle DGF ; and there-

fore as BA to AC , so

is ^c GD to DF : But, by the hypothesis, as BA to AC , so

^d ED to DF ; as therefore ED to DF , so is ^d GD to DF

^e ^{9. 5.} wherefore ED is equal ^e to GD : And DF is common to the

two triangles EDF , GDF ; therefore the two sides ED

DF are equal to the two sides GD , DF ; but the angle ED

is also equal to the angle GDF ; wherefore the base EF is equal

to the base FG ^f, and the triangle EDF to the triangle GDF

and the remaining angles to the remaining angles, each

each, which are subtended by the equal sides: Therefore the

angle DFG is equal to the angle DFE , and the angle at G

the angle at E : But the angle DFG is equal to the angle

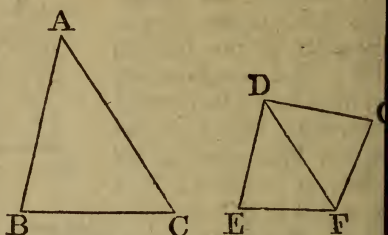
ACB ; therefore the angle ACB is equal to the angle DFE

and the angle BAC is equal to the angle EDF ^g; wherefore

also the remaining angle at B is equal to the remaining angle

at E . Therefore the triangle ABC is equiangular to the triangle

angle DEF . Wherefore, if two triangles, &c. **Q. E. D.**



PROP. VII. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then, if each of the remaining angles be either less, or not less, than a right angle, the triangles shall be equiangular, and have those angles equal about which the sides are proportionals.

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF proportionals, so that AB is to BC, as DE to EF; and in the first case, let each of the remaining angles at C, F be less than a right angle. The triangle ABC is equiangular to the triangle DEF, that is, the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

For, if the angles ABC, DEF be not equal, one of them is greater than the other: Let ABC be the greater, and at the point B, in the straight line

BC, make the angle ABG equal to the angle^a DEF:

and because the angle at B is equal to the angle at E,

and the angle ABG to the angle DEF;

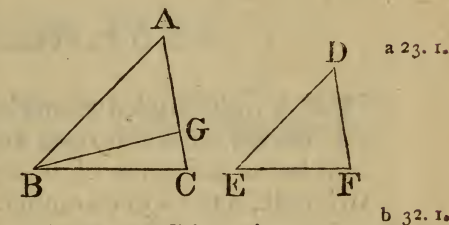
the remaining angle AGB is equal^b to the remaining

angle DFE: Therefore the triangle ABG is equiangular to the triangle DEF; wherefore^c, as AB is to BG, so is DE to EF;

but as DE to EF, so, by hypothesis, is AB to BC: Therefore as AB to BC, so is AB to BG^d; and because AB

is the same ratio to each of the lines BC, BG; BC is equal to BG, and therefore the angle BGC is equal to the angle BGF:

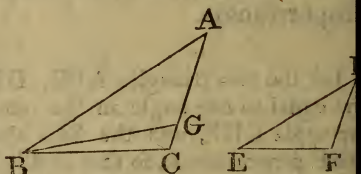
But the angle BCG is, by hypothesis, less than a right angle; therefore also the angle BGC is less than a right angle,



Book VI. and the adjacent angle AGB must be greater than a right angle g . But it was proved that the angle AGB is equal to the angle at F ; therefore the angle at F is greater than a right angle: But, by the hypothesis, it is less than a right angle which is absurd. Therefore the angles ABC , DEF are unequal, that is, they are equal: And the angle at A is equal to the angle at D ; wherefore the remaining angle at C is equal to the remaining angle at F : Therefore the triangle ABC is equiangular to the triangle DEF .

Next, let each of the angles at C , F be not less than a right angle; the triangle ABC is also in this case equiangular to the triangle DEF .

The same construction being made, it may be proved in like manner that BC is equal to BG , and the angle at C equal to the angle BGC : But the angle at C is not less



than a right angle; therefore the angle BGC is not less than a right angle: Wherefore, two angles of the triangle BGC are together not less than two right angles, which is impossible h ; and therefore the triangle ABC may be proved to be equiangular to the triangle DEF , as in the first case.

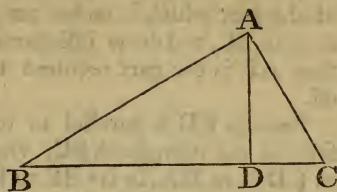
PROP. VIII. THEOR.

IN a right angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right angled triangle, having the right angle BAC ; and from the point A let AD be drawn perpendicular to the base BC : the triangles ABD , ADC are similar to the whole triangle ABC , and to one another.

Because the angle BAC is equal to the angle ADB , each of them being a right angle, and that the angle at B is common

a 32. 1.



b 4. 6.

c 1. def. 6.

non to the two triangles
ABC, ABD; the remain-
ing angle ACB is equal to
the remaining angle BAD^a;
therefore the triangle ABC
is equiangular to the tri-
angle ABD, and the sides
about their equal angles are
proportionals^b; wherefore
the triangles are similar^c: in the like manner it may be de-
monstrated, that the triangle ADC is equiangular and simi-
lar to the triangle ABC: And the triangles ABD, ADC,
being both equiangular and similar to ABC, are equiangular
and similar to each other. Therefore, in a right angled, &c.
Q. E. D.

COR. From this it is manifest, that the perpendicular drawn
from the right angle of a right angled triangle to the base, is
a mean proportional between the segments of the base: and
also that each of the sides is a mean proportional between the
base, and its segment adjacent to that side. For in the tri-
angles BDA, ADC, $BD : DA :: DA : DC$ ^b; and in the
triangles ABC, DBA, $BC : BA :: BA : BD$ ^b; and in
the triangles ABC, ACD, $BC : CA :: CA : CD$ ^b.

PROP. IX. PROB.

FROM a given straight line to cut off any part
required, that is, a part of which the line shall
be a given multiple.

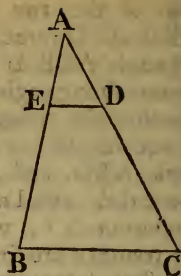
Let AB be the given straight line; it is required to cut off
from it a part of which AB shall be a given multiple.

From the point A draw a straight line AC making any
angle with AB; and in AC take any point D, and take AC

M 4

the

Book VI. the same multiple of AD, that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it: then AE is the part required to be cut off.



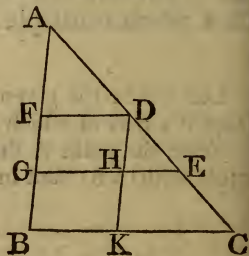
Because ED is parallel to one of the sides of the triangle ABC, viz. to BC,
 a 2. 6. as CD is to DA, so is^a BE to EA; and,
 b 18. 5. by composition^b, CA is to AD, as BA to AE: but CA is a multiple of AD;
 c C. 5. therefore^c BA is the same multiple of AE: whatever part therefore AD is of AC, AE is the same part of AB; wherefore, from the straight line AB the part required is cut off. Which was to be done.

PROP. X. PROB.

TO divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Let AB be the straight line given to be divided, and AC the divided line; it is required to divide AB similarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed so as to contain any angle, and join BC, and through the points D, E, draw^a DF, EG parallel to BC; and through D draw DHK parallel to AB; therefore each of the figures
 a 31. 1. FH, HB, is a parallelogram; wherefore DH is equal^b to FG, and HK to GB: and because
 b 34. 1. HE is parallel to KC, one of the sides of the triangle DKC, as CE to ED, so is^c KH to HD: But
 c 2. 6. KH is equal to BG, and HD to GF; therefore, as CE to ED, so is BG to GF: again, because FD is parallel to EG, one of the sides of the triangle AGE, as ED to DA, so is GF to FA: but it has been proved



that

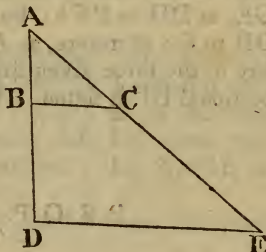
that CE is to ED, as BG to GF; and as ED to DA, so GF to FA: therefore the given straight line AB is divided similarly to AC. Which was to be done. Book VI.

PROP. XI. PROB.

TO find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to AB, AC.

Produce AB, AC to the points D, E; and make BD equal to AC; and having joined BC, through D, draw DE parallel to BC.



a 31. I.

b 2. 6.

Because BC is parallel to DE, side of the triangle ADE, AB is to BD, as AC to CE: but BD is equal to AC; as therefore AB to AC, so is AC to CE. Wherefore to the two given straight lines AB, AC a third proportional CE is found. Which was to be done.

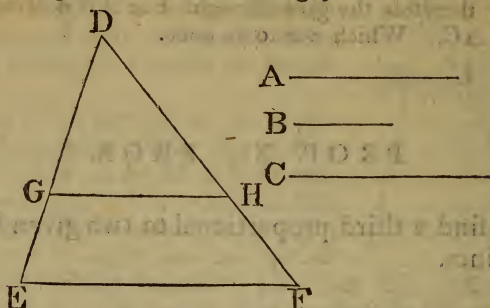
PROP. XII. PROB.

TO find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines; it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF, containing any angle DF; and upon these make DG equal to A, GE equal to B, B,

Book VI. B, and DH equal to C; and having joined GH, draw EF



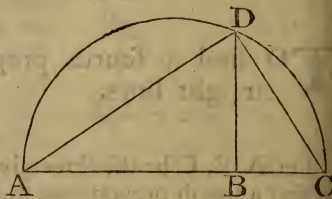
- a 31. 1. parallel ^a to it through the point E: and because GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH to HF ^b; but DG is equal to A, GE to B, and DH to C; therefore, as A is to B, so is C to HF. Wherefore to the three given straight lines, A, B, C a fourth proportional HF is found. Which was to be done.

PROP. XIII. PROB.

TO find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines; it is required to find a mean proportional between them.

- a 11. 1. Place AB, BC in a straight line, and upon AC describe the semicircle ADC, and from the point B draw ^a BD at right angles to AC, and join AD, DC.



- b 31. 3. Because the angle ADC in a semicircle is a right angle ^b, and because in the right angled triangle ADC, DB is drawn from the right angle perpendicular to the base, DB is a mean proportional between AB, BC the segments of the base ^c; therefore between

tween the two given straight lines AB, BC, a mean proportional DB is found. — Which was to be done. Book VI.

PROP. XIV. THEOR.

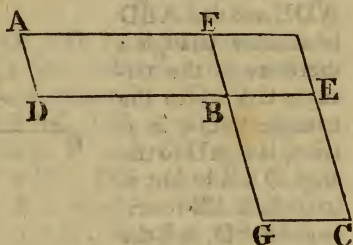
EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: And parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms, which have the angles at B equal, and let the sides DB, BE be placed in the same straight line; wherefore also FB, BG are in one straight line ^a: the sides of the parallelograms AB, BC about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF. a 14. 1.

Complete the parallelogram FE; and because the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to FE ^b: but as AB to FE, so is the base DB to BE ^c; and, as BC to FE, so is the base GB to BF; therefore, as DB to BE, so is GB to BF ^d. Wherefore, the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.

But, let the sides about the equal angles be reciprocally proportional, viz. as DB to BE, so GB to BF; the parallelogram AB is equal to the parallelogram BC.

Because, as DB to BE, so is GB to BF; and as DB to BE, so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is the parallelogram BC to the parallelogram FE; therefore as AB to FE, so BC to FE ^d: wherefore the parallelogram



b 7. 5.

c 1. 6.

d 11. 5.

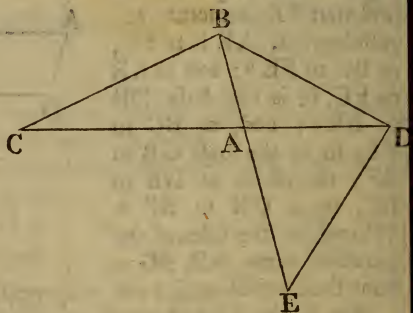
Book VI
e 9. 5. parallelogram AB is equal e to the parallelogram BC. Therefore equal parallelograms, &c. Q. E. D.

PROP. XV. THEOR.

EQUAL triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: And triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the sides about the equal angles of the triangles are reciprocally proportional; that is CA is to AD, as EA to AB.

Let the triangles be placed so that their sides CA, AD be in one straight line; wherefore also EA and AB are in one straight line a; join BD. Because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD, so is triangle EAD to triangle DAB b: but as triangle CAB to triangle BAD, so is the base CA to AD c; and as triangle EAD to triangle DAB, so is the base EA to AB c; as therefore CA to AD, so is EA to AB d; wherefore the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional.



But let the sides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA

EA

EA to AB; the triangle ABC is equal to the triangle ADE.

Having joined BD as before; because, as CA to AD, so is EA to AB; and as CA to AD, so is triangle ABC to triangle BAD^c; and as EA to AB, so is triangle EAD to triangle BAD^c; therefore^d as triangle BAC to triangle BAD, so is triangle EAD to triangle BAD; that is, the triangles BAC, EAD have the same ratio to the triangle BAD: wherefore the triangle BAC is equal^e to the triangle ADE. Therefore equal triangles, &c. Q. E. D.

PROP. XVI. THEOR.

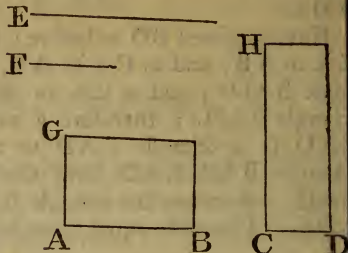
IF four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines, AB, CD, E, F be proportionals, viz. as AB to CD, so E to F; the rectangle contained by AB, F is equal to the rectangle contained by CD, E.

From the points A, C draw^a AG, CH at right angles to AB, CD; and make AG equal to F, and CH equal to E, and complete the parallelograms BG, DH: because, as AB to CD, so is E to F; and that E is equal to CH, and F to AG; AB is^b to CD, as CH to AG: therefore the sides of the parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their sides about equal angles reciprocally proportional, are equal to one another^c; therefore the parallelogram BG is equal to the

Book VI.

the parallelogram DH : and the parallelogram BG is contained by the straight lines AB, F; because AG is equal to F; and the parallelogram DH is contained by CD and E; because CH is equal to E: therefore the rectangle contained by the straight lines AB, F is equal to that which is contained by CD and E.



And if the rectangle contained by the straight lines AB, F be equal to that which is contained by CD, E; these four lines are proportionals, viz. AB is to CD, as E to F.

The same construction being made, because the rectangle contained by the straight lines AB, F is equal to that which is contained by CD, E, and that the rectangle BG is contained by AB, F, because AG is equal to F; and the rectangle DH by CD, E, because CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular: but the sides about the equal angles of equal parallelograms are reciprocally proportional^d: wherefore, as AB to CD, so is CH to AG; and CH is equal to E, and AG to F: as therefore AB is to CD, so E to F. Wherefore, if four, &c. Q. E. D.

PROP. XVII. THEOR.

IF three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: And if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Let the three straight lines A, B, C be proportionals, viz. as A to B, so B to C; the rectangle contained by A, C is equal to the square of B.

Take

Take D equal to B; and because as A to B, so B to C, ^{Book VI.}
 and that B is equal to D; A is a to B, as D to C: but if four ^{a 7. 5.}
 straight lines be proportionals, the rectangle contained by the
 extremes is equal to that which
 is contained by the means ^b:
 therefore the rectangle contained
 by A, C is equal to that contain-
 ed by B, D: but the rectangle
 contained by B, D is the square
 of B; because B is equal to D:
 therefore the rectangle contained
 by A, C is equal to the square of B.

A _____ b 16. 6.
 B _____
 D _____
 C _____

And if the rectangle contained by A, C be equal to the
 square of B; A is to B, as B to C.

The same construction being made, because the rectangle
 contained by A, C is equal to the square of B, and the square
 of B is equal to the rectangle contained by B, D, because B
 is equal to D; therefore the rectangle contained by A, C is
 equal to that contained by B, D: but if the rectangle con-
 tained by the extremes be equal to that contained by the
 means, the four straight lines are proportionals ^b: therefore A
 is to B, as D to C; but B is equal to D; wherefore as A to
 B, so B to C. Therefore, if three straight lines, &c. Q. E. D.

PROP. XVIII. PROB.

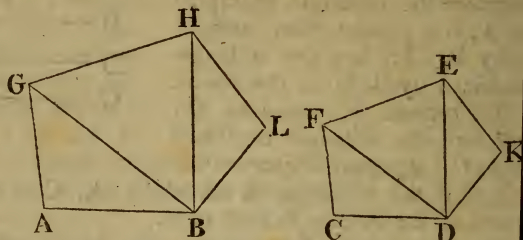
UPON a given straight line to describe a recti-
 lineal figure similar, and similarly situated to
 a given rectilineal figure.

Let AB be the given straight line, and CDEF the given
 rectilineal figure of four sides; it is required upon the given
 straight line AB to describe a rectilineal figure similar, and si-
 milarly situated to CDEF.

Join DF, and at the points A, B in the straight line AB,
 make ^a the angle BAG equal to the angle at C, and the ^{a 23. 1.}
 angle ABG equal to the angle CDF; therefore the remain-
 ing angle CFD is equal to the remaining angle AGB ^b: ^{b 32. 1.}
 wherefore the triangle FCD is equiangular to the triangle
 GAB:

Book VI. GAB: again, at the points G, B in the straight line G

a 23. 1. make the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH: then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE:



for the same reason, the angle ABH is equal to the angle CDE; also the angle at A is equal to the angle at C, and the angle GHB to FED: therefore the rectilineal figure ABHG is equiangular to CDEF: but likewise these figures have their sides about the equal angles proportionals: because the triangles GAB, FCD being equiangular, BA is c to AG, as DC to CF; and because AG is to GB, as CF to FD; and as GB to GH, so, by reason of the equiangular triangles BGH DFE, is FD to FE; therefore, ex æquali d, AG is to GH as CF to FE: in the same manner it may be proved that AB is to BH, as CD to DE: and GH is to HB, as FE to ED c. Wherefore, because the rectilineal figures ABHG CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar to one another e.

e 1. def. 6. Next, Let it be required to describe upon a given straight line AB. a rectilineal figure similar, and similarly situated to the rectilineal figure CDKEF.

Join DE, and upon the given straight line AB describe the rectilineal figure ABHG similar, and similarly situated to the quadrilateral figure CDEF, by the former case; and at the points B, H in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L: and because the figures ABHG CDEF are similar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK: for the same reason

reason the angle ABL is equal to the angle CDK : therefore Book VI.
the five sided figures $AGHLB$, $CFEKD$ are equiangular; and
because the figures $AGHB$, $CFED$ are similar, GH is to
 IB , as FE to ED ; and as HB to HL , so is ED to EK c 4. 6.
therefore, ex æquali ^d, GH is to HL , as FE to EK : for the d 22. 5.
same reason, AB is to BL , as CD to DK : and BL is to LH ,
as DK to KE , because the triangles BLH , DKE are equi-
angular: therefore, because the five sided figures $AGHLB$,
 $CFEKD$ are equiangular, and have their sides about the e-
qual angles proportionals, they are similar to one another:
and in the same manner a rectilineal figure of six or more
sides may be described upon a given straight line similar to
the given, and so on. Which was to be done.

PROP. XIX. THEOR.

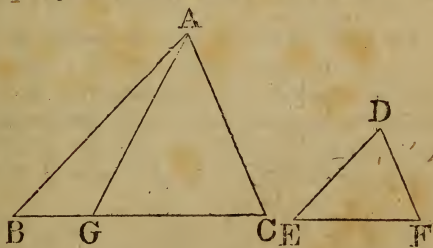
SIMILAR triangles are to one another in the du-
plicate ratio of their homologous sides.

Let ABC , DEF be similar triangles, having the angle B
equal to the angle E , and let AB be to BC , as DE to EF , so
that the side BC is homologous to EF ^a: the triangle ABC a 13. def. 5.
is to the triangle DEF , the duplicate ratio of that which
 BC has to EF .

Take BG a third proportional to BC , EF ^b, so that BC is b 11. 6.
to EF , as EF to BG , and join GA : then, because as AB to

C , so DE to EF ;
alternately ^c, AB is
to DE , as BC to
 EF : but as BC to
 EF , so is EF to
 BG ; therefore ^d as
 AB to DE , so is EF
to BG : wherefore

the sides of the tri-
angles ABG , DEF which are about the equal angles, are reci-
procally proportional: but triangles which have the sides a-
bout



c 16. 5.

d 11. 5.

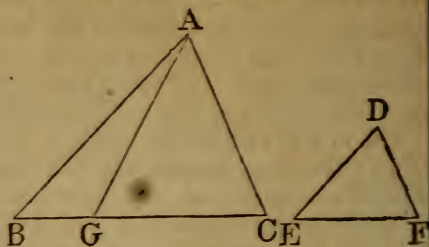
N

bout

Book VI. about two equal angles reciprocally proportional are equal to one another: therefore the triangle

e 15. 6.

ABG is equal to the triangle DEF; and because as BC is to EF, so EF to BG; and that if three straight lines be proportionals, the first has to the



f 1. 6.

third the duplicate ratio of that which it has to the second; BC therefore has to BG the duplicate ratio of that which BC has to EF: but as BC to BG, so is the triangle ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: but the triangle ABG is equal to the triangle DEF; wherefore also the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore similar triangles, &c. Q. E. D.

COR. From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar, and similarly described triangle upon the second.

PROP. XX. THEOR.

SIMILAR polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHLK be similar polygons, and let AE be the homologous side to FG: the polygons ABCDE, FGHLK may be divided into the same number of similar triangles, whereof each to each has the same ratio which the polygons have; and the polygon ABCDE has to the polygon FGHLK the duplicate ratio of that which the side AE has to the side FG.

Join BE, EC, GL, LH: and because the polygon ABCDE is similar to the polygon FGHLK, the angle BAE is equal to the angle GFL, and BA is to AE, as GF to FL: wherefore, because the triangles ABE, FGL have an angle in one equal to an angle in the other, and their sides about

a 1. def. 6.

the

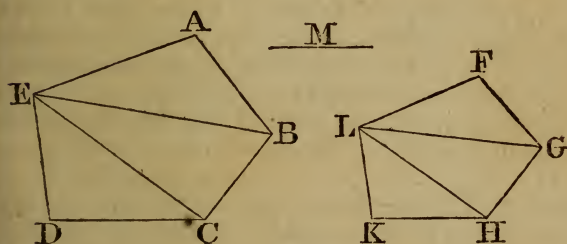
Book VI.

b 6. 6.

c 4. 6.

d 22. 5.

These equal angles proportionals, the triangle ABE is equiangular^b, and therefore similar to the triangle FGL^c; wherefore the angle ABE is equal to the angle FGL: and, because the polygons are similar, the whole angle ABC is equal to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH: and because the triangles ABE, FGL are similar, EB is to BA, as LG to GF^a; and also, because the polygons are similar, AB is to BC, as FG to GH^a; therefore, ex æquali^d, EB is to BC as LG to GH; that is, the sides about the equal angles EBC, LGH are proportionals; therefore^b the triangle EBC is equiangular to the triangle LGH, and similar to it^c. For the same reason, the triangle ECD likewise is similar to the triangle LHK; therefore the similar polygons ABCDE, FGHLK are divided into the same number of similar triangles.



Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE has to the polygon FGHLK the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, BE has to FGL the duplicate ratio^e of that which the side BE has to the side GL: for the same reason, the triangle EC has to GLH the duplicate ratio of that which BE has to GL: therefore, as the triangle ABE to the triangle FGL, so is the triangle BEC to the triangle GLH. Again, because the triangle EBC is similar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH: for the same reason, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH: as therefore the triangle EBC to the triangle LGH, so is the triangle ECD to the triangle LHK: but it has been proved, that the triangle EBC is likewise to the

e 19. 6.

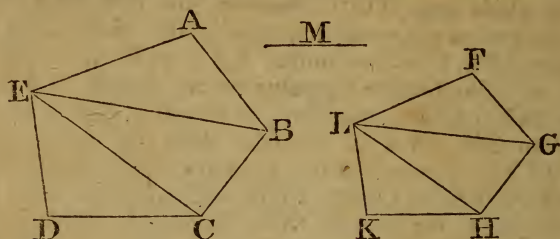
f 11. 5.

N 2

triangle

Book VI.

triangle LGH, as the triangle ABE to the triangle FGL. Therefore; as the triangle ABE is to the triangle FGL, so is triangle EBC to triangle LGH, and triangle ECD to triangle LHK : and therefore, as one of the antecedents to one



of the consequents, so are all the antecedents to all the consequents g. Wherefore, as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHLK but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the side AB has to the homologous side FG. Therefore also the polygon ABCDE has to the polygon FGHLK the duplicate ratio of that which AB has to the homologous side FG. Wherefore similar polygons, & Q. E. D.

COR. 1. In like manner it may be proved, that similar four sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides, and has already been proved in triangles. Therefore, universally similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

COR. 2. And if to AB, FG, two of the homologous sides h r r. def. 5. a third proportional M be taken, AB has ^h to M the duplicate ratio of that which AB has to FG : but the four sided figure or polygon upon AB has to the four sided figure or polygon upon FG likewise the duplicate ratio of that which AB has to FG : therefore, as AB is to M, so is the figure upon AB to the figure upon FG, which was also proved in triangles i Cor. 19. 6. angles i. Therefore, universally, it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

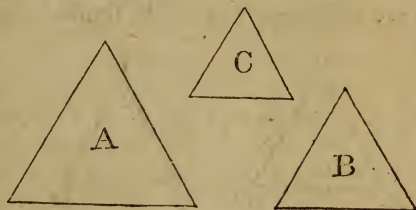
COR. 3. Because all squares are similar figures, the ratio of any two squares to one another is the same with the duplicate ratio of their sides ; and hence, also, any two similar rectilineal figures are to one another as the squares of their homologous sides.

PROP. XXI. THEOR.

RECTILINEAL figures which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures A, B be similar to the rectilineal figure C: The figure A is similar to the figure

Because A is similar to C, they are equiangular, and also have their sides about the equal angles proportionals^a. Again, ^a 1. def. 6. Because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals^a: Therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular^b, ^b 1. Ax. 1. and have their sides about the equal angles proportionals^c, ^c 11. 5. Therefore A is similar^a to B. Q. E. D.



PROP. XXII. THEOR.

IF four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD

N 3

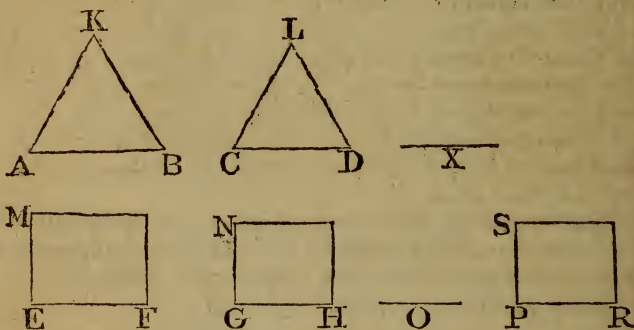
let

Book VI. let the similar rectilineal figures KAB, LCD be similarly described; and upon EF GH the similar rectilineal figures MF NH in like manner: the rectilineal figure KAB is to LCD as MF to NH.

- a 11. 6. To AB, CD take a third proportional^a X; and to EF GH a third proportional O: and because AB is to CD, as EF to GH, and that CD is^b to X, as GH to O; wherefore
 b 11. 5. ex æquali^c, as AB to X, so EF to O: but as AB to X, so
 c 22 5. is^d the rectilineal KAB to the rectilineal LCD, and as EF to O, so is^d the rectilineal MF to the rectilineal NH: therefore
 d 2. cor. as KAB to LCD, so^b is MF to NH.
 20. 6.

And if the rectilineal KAB be to LCD, as MF to NH the straight line AB is to CD, as EF to GH.

- e 12. 6. Make^e as AB to CD, so EF to PR, and upon PR describe
 f 18. 6. f the rectilineal figure SR similar and similarly situated to



- either of the figures MF, NH: then, because as AB to CD so is EF to PR, and that upon AB, CD are described the similar and similarly situated rectilineals KAB, LCD, and upon EF, PR, in like manner, the similar rectilineals MF, SR. KAB is to LCD, as MF to SR; but, by the hypothesis KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the same ratio to each of the two NH, SR, these are equal^g to one another: they are also similar, and similarly situated; therefore GH is equal to PR: and because as AB to CD, so is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four straight lines, &c. Q. E. D.

PROP. XXIII. THEOR.

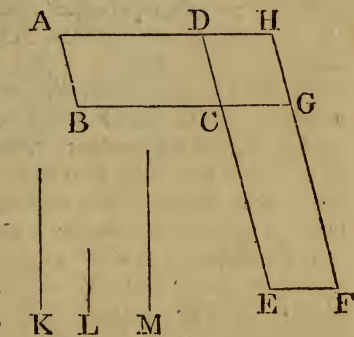
EQUIANGULAR parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: the ratio of the parallelogram AC to the parallelogram CF, is the same with the ratio which is compounded of the ratios of their sides.

Let BC, CG be placed in a straight line; therefore DC and CE are also in a straight line ^a; and complete the parallelogram DG; and, taking any straight line K, make ^b as BC to CG, so K to L; and as DC to CE, so make ^b L to M: therefore the ratios of K to L, and L to M, are the same with the ratios of the sides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is said to be compounded ^c of the ratios of K to L, and L to M; where- ^c 10. def. 5.

fore also K has to M the ratio compounded of the ratios of the sides: and because as BC to CG, so is the parallelogram AC to the parallelogram CH ^d; but as BC to CG, so is K to L; therefore K is ^e to L, as the parallelogram AC to the parallelogram CH: again, because as DC to CE, so is the parallelogram CH to the parallelogram CF; but as DC to CE, so is L to M; wherefore L is

^e to M, as the parallelogram CH to the parallelogram CF: therefore, since it has been proved, that as K to L, so is the parallelogram AC to the parallelogram CH; and as L to M, so the parallelogram CH to the parallelogram CF; ex æquali ^f, K is to M, as the parallelogram AC to the parallelogram ^f 22. 5]



d 1. 6.

e 11. 5.

N 4

CF:

Book VI. CF : but K has to M the ratio which is compounded of the ratios of the sides ; therefore also the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides. Wherefore equiangular parallelogram &c. $Q. E. D.$

PROP. XXIV. THEOR.

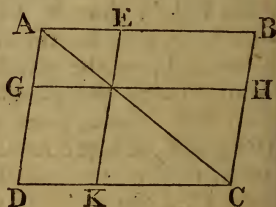
THE parallelograms about the diameter of any parallelogram, are similar to the whole, and to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC ; and EG, HK the parallelograms about the diameter. The parallelograms EG, HK are similar both to the whole parallelogram $ABCD$, and to one another.

Because DC, GF are parallels, the angle ADC is equal to the angle AGF : for the same reason, because BC, EF are parallels, the angle ABC is equal to the angle AEF : and each of the angles BCD, EFG is equal to the opposite angle DAB ^b, and therefore are equal to one another, wherefore the parallelograms $ABCD, AEFG$ are equiangular : and because the angle ABC is equal to the angle AEE , and the angle BAC common to the two triangles BAC, EAF , they are equiangular to one another ; therefore ^c as AB to BC , so is AE to EF : and because the opposite sides of parallelograms are equal

to one another ^b, AB is ^d to AD , as AE to AG ; and DC to CB , as GF to FE ; and also CD to DA , as FG to GA : therefore

the sides of the parallelograms $ABCD, AEFG$ about the equal angles are proportionals ; and they are therefore similar to one another ^e : for the same reason, the parallelogram $ABCD$ is similar to the parallelogram $FHCK$. Wherefore each of the parallelograms, GE, KH is similar to DB : but rectilineal figures which are similar to the same rectilineal figure,



^e 1. def. 6.

figure, are also similar to one another ^f; therefore the parallelogram GE is similar to KH. Wherefore the parallelograms, ^{Book VI.}
 &c. Q. E. D. ^{f 21. 6.}

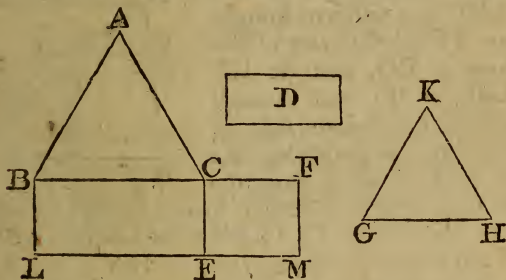
PROP. XXV. PROB.

TO describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be described is required to be similar, and D that to which it must be equal. It is required to describe a rectilineal figure similar to ABC, and equal to D.

Upon the straight line BC describe ^a the parallelogram BE ^{a cor. 45. 1.} equal to the figure ABC; also upon CE describe ^a the parallelogram CM equal to D, and having the angle FCE equal to the angle CBL: therefore BC and CF are in a straight line ^b, as also LE and EM between BE and BF find ^c a mean proportional GH, and upon GH describe ^d the rectili-

^b { 29. 1.
 14. 1.
^c 13. 6.
^d 18. 6.



al figure KGH similar and similarly situated to the figure BC: and because BC is to GH as GH to CF, and if three straight lines be proportional, as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the second; therefore as BC to CF, so is the rectilineal figure ABC to KGH: but as BC to CF, so is the parallelogram BE to the parallelogram EF: therefore

^e 2. cor.
 20. 6.
^f 1. 6.

as

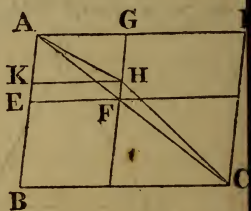
Book VI. as the rectilineal figure ABC is to KGH, so is the parallelogram BE to the parallelogram EF ^g 11. 5. : and the rectilineal figure ABC is equal to the parallelogram BE; therefore the rectilineal figure KGH is equal ^h 14. 5. to the parallelogram EF: but EF is equal to the figure D; wherefore also KGH is equal to D; and it is similar to ABC. Therefore the rectilineal figure KGH has been described similar to the figure ABC, and equal to D. Which was to be done.

PROP. XXVI. THEOR.

IF two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let the parallelograms ABCD, AEFG be similar and similarly situated, and have the angle DAB common. ABCD and AEFG are about the same diameter.

For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC: therefore the parallelograms ABCD, AKHG being about the

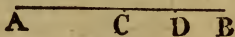


For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC: therefore the parallelograms ABCD, AKHG being about the same diameter, they are similar to one another ^a 24. 6. : wherefore as DA to AB, so is ^b GA to AK: but because ABCD and AEFG are similar parallelograms, as DA is to AB, so is GA to AE; therefore ^c as GA to AE, so GA to AK: wherefore GA has the same ratio to each of the straight lines AE, AK; and consequently AK is equal ^d 9. 5. to AE, the less to the greater, which is impossible: therefore ABCD and AKHG are not about the same diameter; wherefore ABCD and AEFG must be about the same diameter. Therefore, two similar, &c. Q. E. D.

PROP. XXVII. THEOR.

OF all the rectangles contained by the segments of a given straight line, the greatest is the square which is described on half the line.

Let AB be a given straight line, which is bisected in C ; and let D be any point in it, the square on AC is greater than the rectangle AD, DB .



For since the straight line AB is divided into two equal parts in C , and into two unequal parts in D , the rectangle contained by AD and DB , together with the square of CD , is equal to the square of AC ^a. The square of AC is therefore greater than the rectangle AD, DB . Therefore, &c.
E. D.

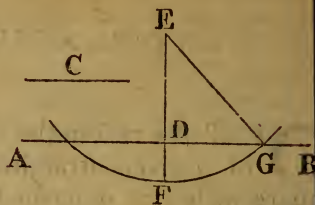
PROP. XXVIII. PROB.

TO divide a given straight line, so that the rectangle contained by its segments may be equal to a given space; but that space must not be greater than the square of half the given line.

Let AB be the given straight line, and let the square upon the given straight line C be the space to which the rectangle contained by the segments of AB must be equal, and this square, by the determination, is not greater than that upon half the straight line AB .

Bisect AB in D , and if the square upon AD be equal to the square upon C , the thing required is done: But if it be not equal

Book VI. equal to it, AD must be greater than C, according to the de-
 termination: Draw DE at
 right angles to AB, and
 make it equal to C; produce
 ED to F, so that EF be e-
 qual to AD or DB, and from
 the centre E, at the distance
 EF, describe a circle meet-
 ing AB in G. Join EG;
 and because AB is divided
 equally in D, and unequal-



a 5. 2. DG is equal ^a to the square of DB, that is, of EF or EG;
 but the squares of ED, DG are also equal to the square of EG;
 b 47. 1. EG ^b; therefore the rectangle AG, GB, together with the
 square of DG, is equal to the squares of ED and DG; take
 away the square of DG from each of these equals; therefor
 the remaining rectangle AG, GB is equal to the square of
 ED, that is, of C: Wherefore the given line AB is divided
 in G, so that the rectangle contained by the segments AG,
 GB is equal to the square upon the given straight line C.
 Which was to be done.

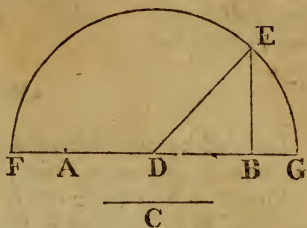
PROP. XXIX. PROB.

TO produce a given straight line, so that the
 rectangle contained by the segments between
 the extremities of the given line and the point to
 which it is produced, may be equal to a given space

Let AB be the given straight line, and let the square upon
 the given straight line C be the space to which the rectangle
 under the segments of AB produced, must be equal.

Bisect AB in D, and draw BE at right angles to it, so that
 BE be equal to C; and having joined DE, from the centre D
 at the distance DE describe a circle meeting AB produced in
 G.

7. And because AB is bisected in D, and produced to G, the rectangle AG, GB together with the square of DB is equal ^a to the square of DG, or of DE, that is, to the squares of EB, BD ^b. From each of these equals take the square of DB; therefore the remaining rectangle AG, GB is equal to the square of BE, that is, to the square upon C. Wherefore the line AB is produced to G, so that the rectangle contained by the segments AG, GB of the line produced is equal to the square of C. Which was to be done.



a 6. 2.

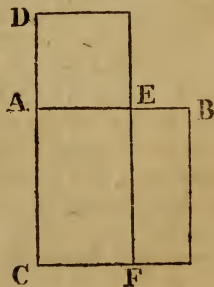
b 47. 1.

PROP. XXX. PROB.

TO cut a given straight line in extreme and mean ratio.

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.

Upon AB describe ^a the square BC, and produce CA to D, so that the rectangle CD, DA may be equal to the square CB ^b. Take AE equal to AD, and complete the rectangle DF under DC and AE, or under DC and DA. Then, because the rectangle CD, DA is equal to the square CB, the rectangle DF is equal to CB. Take away the common part CE from each, and the remainder FB is equal to the remainder DE. But FB is the rectangle contained by FE and EB, that is, by AB and BE; and DE is the square upon AE; therefore AE is a mean proportional between AB and BE ^c, or AB is to AE as AE to EB. But AB is greater than AE;



c 17. 6.

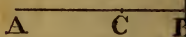
wherefore

Book VI. wherefore AE is greater than EB^e : Therefore the straight
^{e 14. 5.} line AB is cut in extreme and mean ratio in E^f . Which
^{f 3. def. 6.} was to be done.

Otherwise,

Let AB be the given straight line; it is required to cut
 in extreme and mean ratio.

Divide AB in the point C , so that the rectangle contained
 by AB, BC be equal to the square of
^{g 11. 2.} AC^g : Then, because the rectangle
 AB, BC is equal to the square of AC ,
^{h 17. 6.} as BA to AC , so is AC to CB^h :
 Therefore AB is cut in extreme and mean ratio in C .
 Which was to be done.

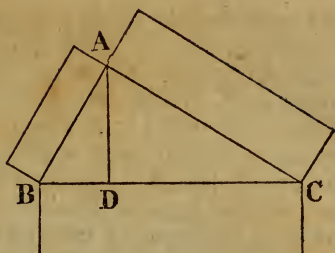


PROP. XXXI. THEOR.

IN right angled triangles, the rectilineal figure described upon the side opposite to the right angle is equal to the similar, and similarly described figures upon the sides containing the right angle.

Let ABC be a right angled triangle, having the right angle BAC : The rectilineal figure described upon BC is equal to the similar, and similarly described figures upon BA, AC .

Draw the perpendicular AD ; therefore, because in the
 right angled triangle ABC , AD is drawn from the right angle
^{a 8. 6.} at A perpendicular to the base BC , the triangles ABD, ADC
 are similar to the whole triangle ABC , and to one another^a,
^{b 4. 6.} and because the triangle ABC is similar to ADB , as CB to
 BA , so is BA to BD^b ; and because these three straight lines
 are proportionals, as the first to the third, so is the figure upon
 the first to the similar, and similarly described figure upon the
^{c 2. Cor.} second^c: Therefore as CB to BD , so is the figure upon
^{20. 6.} CB



e 24. 5.

f A. 5.

PRO P. XXXII. T H E O R.

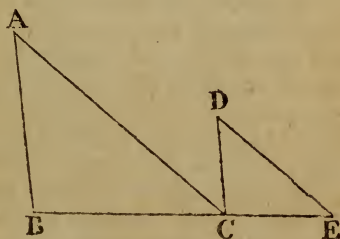
If two triangles which have two sides of the one proportional to two sides of the other, be joined in one angle, so as to have their homologous sides parallel to one another; the remaining sides shall be in a straight line.

Let ABC, DCE be two triangles which have the two sides BA, AC proportional to the two CD, DE, viz. BA to AC, CD to DE; and let AB be parallel to DC, and AC to CE, BC and CE are in a straight line.

Because AB is parallel to DC, and the straight line AC meets them, the alternate angles BAC, ACD are equal^a; for the same reason, the angle CDE is equal to the angle ACD; therefore also BAC is equal to CDE: And because the triangles ABC, DCE have

a 29. 1.

the angle at A equal to the angle at D, and the sides about these angles proportionals, viz. BA to AC, CD to DE, the triangle ABC is equiangular to DCE: Therefore the angle ABC is equal to the angle DCE: And the angle BAC was pro-



b 6. 5.

ved

Book VI. ved to be equal to $\angle ACD$: Therefore the whole angle $\angle AC$
 is equal to the two angles $\angle ABC, \angle BAC$; add the comm
 angle $\angle ACB$, then the angles $\angle ACE, \angle ACB$ are equal to t
 angles $\angle ABC, \angle BAC, \angle ACB$: But $\angle ABC, \angle BAC, \angle ACB$ are equ
 c 32. 1. to two right angles c ; therefore also the angles $\angle ACE, \angle AC$
 are equal to two right angles: And since at the point C ,
 the straight line AC , the two straight lines BC, CE , whi
 are on the opposite sides of it, make the adjacent ang
 d 14. 1. $\angle ACE, \angle ACB$ equal to two right angles; therefore^d BC a
 CE are in a straight line. Wherefore, if two triangles, &
 $Q. E. D.$

PROP. XXXIII. THEOR.

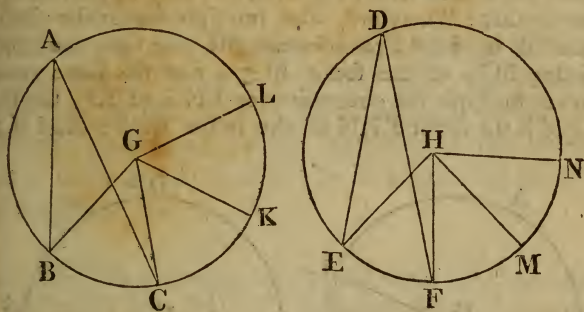
IN equal circles, angles, whether at the centres
 circumferences, have the same ratio which th
 circumferences on which they stand have to one
 another: So also have the sectors.

Let ABC, DEF be equal circles; and at their centres t
 angles BGC, EHF , and the angles BAC, EDF at their ci
 cumferences; as the circumference BC to the circumfere
 EF , so is the angle BGC to the angle EHF , and the ang
 BAC to the angle EDF ; and also the sector BGC to th
 sector EHF .

Take any number of circumferences CK, KL , each equal
 BC , and any number whatever FM, MN each equal to EF
 and join GK, GL, HM, HN . Because the circumfere
 BC, CK, KL are all equal, the angles BGC, CGK, KGL a
 a 27. 3. also all equal^a: Therefore what multiple soever the circum
 ference BL is of the circumference BC , the same multiple
 the angle BGL of the angle BGC : For the same reaso
 whatever multiple the circumference EN is of the circum
 ference EF , the same multiple is the angle EHN of the ang
 EHF : And if the circumference BL be equal to the circum
 ference EN , the angle BGL is also equal^a to the angle EHN
 and if the circumference BL be greater than EN , likewise th
 angle BGL is greater than EHN ; and if less, less: Ther
 being then four magnitudes, the two circumferences BC, EN
 and

and the two angles BGC, EHF, of the circumference BC, and the angle BGC, have been taken any equimultiples whatever, viz. the circumference BL, and the angle BGL; and of the circumference EF, and of the angle EHF, any equimul-

Book VI.



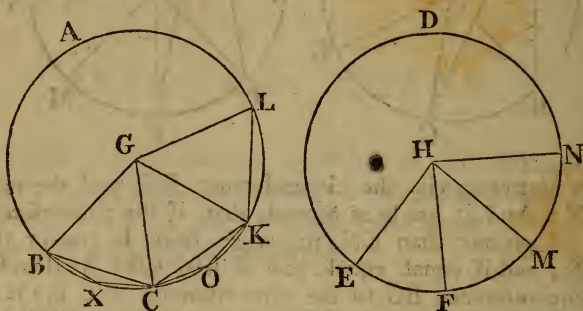
les whatever, viz. the circumference EN, and the angle HN: And it has been proved, that, if the circumference L be greater than EN, the angle BGL is greater than HN; and if equal, equal; and if less, less: As therefore the circumference BC to the circumference EF, so^b is the angle BGC to the angle EHF: But as the angle BGC is to the angle EHF, so is^c the angle BAC to the angle EDF, for each is double of each^d: Therefore, as the circumference BC to EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.

Also, as the circumference BC to EF, so is the sector BGC to the sector EHF. Join BC, CK, and in the circumferences BC, CK take any points X, O, and join BX, XC, CO, OK: Then, because in the triangles GBC, GCK the two sides BG, GC are equal to the two CG, GK, and that they contain equal angles; the base BC is equal^e to the base CK, and the angle GBC to the angle GCK: And because the circumference BC is equal to the circumference CK, the remaining part of the whole circumference of the circle ABC, is equal to the remaining part of the whole circumference of the same circle: Wherefore the angle BXC is equal to the angle OKA; and the segment BXC is therefore similar to the segment COK^f; and they are upon equal straight lines BC, CK: At similar segments of circles upon equal straight lines, are equal^g to one another: Therefore the segment BXC is equal

O

to

Book VI. to the segment COK : And the triangle BGC is equal to the triangle CGK ; therefore the whole, the sector BGC, is equal to the whole, the sector CGK : For the same reason, the sector KGL is equal to each of the sectors BGC, CGK : In the same manner, the sectors EHF, FHM, MHN may be proved equal to one another : Therefore, what multiple soever the circumference BL is of the circumference BC, the same multiple is the sector BGL of the sector BGC : For the same reason whatever multiple the circumference EN is of EF, the same multiple is the sector EHN of the sector EHF : And if the



circumference BL be equal to EN, the sector BGL is equal to the sector EHN ; and if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN and if less, less : Since, then, there are four magnitudes, the two circumferences BC, EF, and the two sectors BGC, EHF and of the circumference BC, and sector BGC, the circumference BL and sector BGL are any equal multiples whatever and of the circumference EF, and sector EHF, the circumference EN and sector EHN, are any equimultiples whatever and that it has been proved, if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN ; and if equal, equal ; and if less, less. Therefore ^{b 5. def. 5.} the circumference BC is to the circumference EF, so is the sector BGC to the sector EHF. Wherefore, in equal circles &c. Q. E. D.

PROP. B. THEOR.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD ; the rectangle BA, AC is equal to the rectangle BD, DC , together with the square of AD .

Describe the circle ^a ACB about the triangle, and produce ^{a 5. 4.}

AD to the circumference in E ,

and join EC : Then because the

angle BAD is equal to the angle

CAE , and the angle ABD to

angle ^b AEC , for they are in

the same segment; the triangles

ABD, AEC are equiangular to

another: Therefore as BA

AD , so is ^c EA to AC , and

consequently the rectangle $BA,$

AD is equal ^d to the rectangle

EA, AD , that is ^e, to the rect-

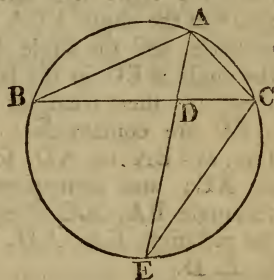
angle ED, DA , together with the square of AD : But the

rectangle ED, DA is equal to the rectangle ^f BD, DC . ^{f 35. 3.}

Therefore the rectangle BA, AC is equal to the rectangle

BD, DC , together with the square of AD . Wherefore, if an

angle, &c. $Q. E. D.$



^b 21. 3.

^c 4. 6.

^d 16. 6.

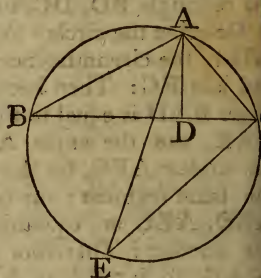
^e 3. 2.

PROP. C. THEOR.

IF from any angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the base BC ; the rectangle BA, AC is equal to the rectangle contained by AD and the diameter of the circle described about the triangle.

Describe ^a the circle ACB about the triangle, and draw its diameter AE , and join EC : Because the right angle BDA is equal ^b to the angle ECA in a semicircle, and the angle ABD to the angle AEC in the same segment ^c; the triangles ABD, AEC are equiangular: Therefore, as ^d BA to AD , so is EA to AC ; and consequently the rectangle BA, AC is equal ^e to the rectangle EA, AD . If therefore, from an angle, & $Q. E. D.$



PROP. D. THEOR.

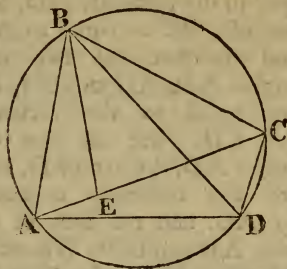
See N.

THE rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let $ABCD$ be any quadrilateral inscribed in a circle, and join AC, BD ; the rectangle contained by AC, BD is equal to the sum of the rectangles contained by AB, CD and AD, BC .

to the two rectangles contained by AB, CD, and by AD, Book VI.
BC.

Make the angle ABE equal to the angle DBC; add to each of these the common angle EBD, then the angle ABD is equal to the angle EBC: And the angle BDA is equal^a to the angle BCE, because they are in the same segment; therefore the triangle ABD is equiangular to the triangle BCE: Wherefore^b as BC is to CE, so is BD to DA; and consequently the rectangle BC, AD is equal^c to the rectangle BD, CE: Again, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD: As therefore BA is to AE, so is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE: But the rectangle BC, AD has been shewn equal to the rectangle BD, CE; therefore the whole rectangle AC, BD^d is equal to the rectangle AB, DC, together with the rectangle AD, BC. Therefore the rectangle, &c. Q. E. D.



b 4. 6.

c 16. 6.

d 1. 2.

PROP. E. THEOR.

IF a segment of a circle be bisected, and from the extremities of the base of the segment, and from the point of bisection straight lines be inflected to any point in the circumference, the sum of the two lines drawn from the extremities of the base will have to the line drawn from the point of bisection the same ratio which the base of the segment has to the base of half the segment.

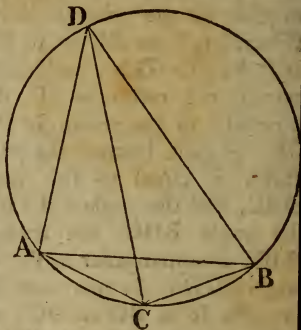
Let ABD be a circle, of which AB is a segment bisected in C, and from A, C and B to D, any point whatever in the circumference,

O 3

circumference,

Book VI. circumference, let AD, CD, BD be drawn; the sum of the two lines AD and DB has to DC the same ratio that BA has to AC.

For since ACBD is a quadrilateral inscribed in a circle, of which the diagonals are AB and CD, the rectangles AD, CB and DB, AC are together equal to the rectangle^a AB, CD. But the rectangle AD, BC is equal to the rectangle AD, AC, because BC is equal to AC; and therefore the two rectangles AD, AC and BD, AC are equal to the rectangle AB, CD. But the two rectangles AD, AC, and BD, AC, are the rectangle contained by AC, and the sum of the lines AD and DB^b. Wherefore the rectangle contained by AC and the sum of the lines AD, DB is equal to the rectangle AB, CD; and because the sides of equal rectangles are reciprocally proportional^c, the sum of AD and DB is to DC as AB to AC. Q. E. D.



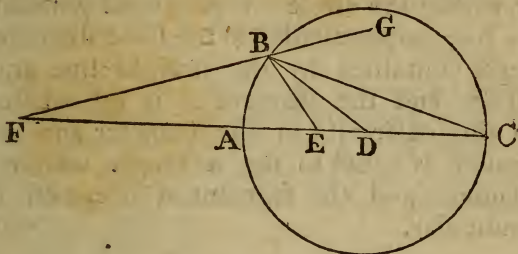
PROP. F. THEOR.

IF two points be taken in the diameter of a circle such that the rectangle contained by the segments intercepted between them and the centre of the circle be equal to the square of the semidiameter; and if from these points two straight lines be inflected to any point whatsoever in the circumference of the circle, the ratio of the lines inflected will be the same with the ratio of the segments intercepted between the two first mentioned points and the circumference of the circle.

Let ABC be a circle, of which the centre is D, and in DA produced, let the points E and F be such that the rectangle ED

FD, DF is equal to the square of AD; from E and F to any Book VI.
 point B in the circumference, let EB, FB be drawn;
 $FB : BE :: FA : AE$.

Join BD, and because the rectangle FD, DE is equal to the
 square of AD, that is, of DB, $FD : DB :: DB : DE^a$. The a 17. 6.



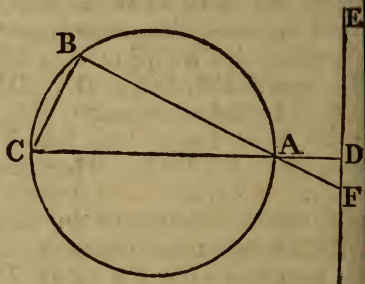
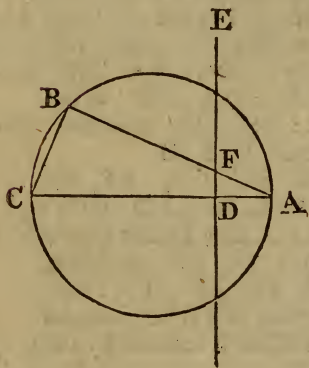
Two triangles, FDB, BDE have therefore the sides propor-
 tional that are about the common angle D; therefore they
 are equiangular ^b, the angle DEB being equal to the angle ^{b 6. 6.}
 DBF, and DBE to DFB. Now, since the sides about these
 equal angles are also proportional ^c, $FB : BD :: BE : ED$, and ^{c 4. 6.}
 alternately ^d, $FB : BE :: BD : ED$, or $FB : BE :: AD : DE$. ^{d 16. 5.}
 But because $FD : DA :: DA : DE$, by division ^e, $FA : DA ::$ ^{e 17. 5.}
 $AE : ED$, and alternately ^e, $FA : AE :: DA : ED$. Now,
 has been shewn that $FB : BE :: AD : DE$, therefore ^f $f 11. 5.$
 $FB : BE :: FA : AE$. Therefore, &c. Q. E. D.

COR. If AB be drawn, because $FB : BE :: FA : AE$, the
 angle FBE is bisected ^g by AB. Also, since $FD : DC ::$ ^{g 3. 6.}
 $FC : DE$, by composition ^h, $FC : DC :: CE : ED$, and since it ^{h 18. 5.}
 has been shewn that $FA : AD (DC) :: AE : ED$, therefore,
 ex æquo, $FA : AE :: FC : CE$. But $FB : BE :: FA : AE$,
 therefore, $FB : BE :: FC : CE$ ^f; so that if FB be produced
 to G, and if BC be drawn, the angle EBG is bisected by the
 line BC ^k. k A. 6.

PROP. G. THEOR.

IF from the extremity of the diameter of a circle a straight line be drawn in the circle, and if either within the circle or produced without it, it meet a line perpendicular to the same diameter, the rectangle contained by the straight line drawn in the circle, and the segment of it intercepted between the extremity of the diameter and the perpendicular, is equal to the rectangle contained by the diameter and the segment of it cut off by the perpendicular.

Let ABC be a circle, of which AC is a diameter, let DE be perpendicular to the diameter AC, and let AB meet DE in F; the rectangle BA, AF is equal to the rectangle CA, AD. Join BC, and because ABC is an angle in a semicircle



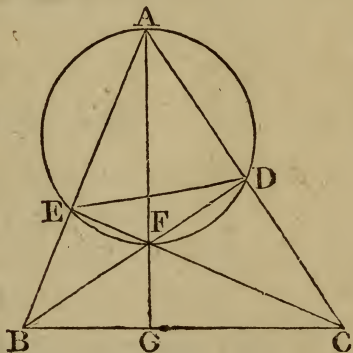
- a 31. 3. it is a right angle^a: Now, the angle ADF is also a right
 b Hyp. angle^b; and the angle BAC is either the same with DAF,
 or vertical to it; therefore the triangles ABC, ADF are
 c 4. 6. equiangular, and $BA:AC::AD:AF$ ^c; therefore also the
 rectangle BA, AF, contained by the extremes, is equal to
 d 16. 6. the rectangle AC, AD contained by the means^d. If therefore,
 &c. Q. E. D.

PROP.

PROP. H. THEOR.

THE perpendiculars drawn from the three angles of any triangle to the opposite sides intersect one another in the same point.

Let ABC be a triangle, BD and CE two perpendiculars intersecting one another in F ; let AF be joined, and, produced if necessary, let it meet BC in G , AG is perpendicular to BC . Join DE , and about the triangle AEF let a circle be described, AEF ; then, because AEF is a right angle, the circle described about the triangle AEF will have AF for its diameter^a. In the same manner, the circle described about the triangle DFC has DF for its diameter; therefore the points E , F and D are in the circumference of the same circle. But because the angle EFB is equal to the angle DFC ^b, and also the angle BEF to the angle EDF , being both right angles, the triangles BEF and CDF are equiangular, and therefore $BF:EF::CF:FD$,^c alternately^d, $BF:FC::EF:FD$. Since, then, the sides about the equal angles BFC , EFD are proportionals, the triangles BFC , EFD are also equiangular^e; wherefore the angle CBF is equal to the angle EDF . But EDF is equal to EAF , because they are angles in the same segment^f; therefore the angle EAF is equal to the angle FCG : Now, the angles AEF , CFG are also equal, because they are vertical angles; therefore the remaining angles AEF , FGC are also equal^g: But AEF is a right angle, therefore FGC is a right angle, and AG is perpendicular to BC . Q. E. D.



a 37. 3.

b 15. 1.

c 4. 6.

d 16. 5.

e 6. 6.

f 21. 3.

g 32. 1.

BOOK THE FIRST

THE first part of this book contains the definitions, axioms, and postulates of geometry. It also contains the first four propositions of the first book of Euclid's Elements.

The second part of this book contains the definitions, axioms, and postulates of geometry. It also contains the first four propositions of the first book of Euclid's Elements.



The third part of this book contains the definitions, axioms, and postulates of geometry. It also contains the first four propositions of the first book of Euclid's Elements.

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E L E M E N T S

O F

G E O M E T R Y.

B O O K VII.

D E F I N I T I O N S.

I.

A straight line is perpendicular, or at right angles to a Book VII.
plane, when it makes right angles with every straight 
line meeting it in that plane.

II.

plane is perpendicular to a plane, when the straight lines
drawn in one of the planes perpendicularly to the common
section of the two planes, are perpendicular to the other
plane.

III.

the inclination of a straight line to a plane is the acute angle
contained by that straight line and another drawn from
the point in which the first line meets the plane, to the
point

Book VII.

point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.

IV.

The inclination of a plane to a plane is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.

V.

Two planes are said to have the same, or a like inclination one another, which two other planes have, when the same angles of inclination are equal to one another.

VI.

Parallel planes are such as do not meet one another though produced.

VII.

A Solid is that which has length, breadth, and thickness.

VIII.

See N. A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, one point.

IX.

See N. Similar solid figures are such as are contained by the same number of similar planes similarly situated, and having like inclinations to one another.

X.

A pyramid is a solid figure contained by planes that are cut off by one plane and a point above it in which they meet.

XI.

prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others parallelograms.

XII.

parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

XIII.

cube is a solid figure contained by six equal squares.

XIV.

sphere is a solid figure described by the revolution of a semicircle about a diameter, which remains unmoved.

XV.

the axis of a sphere is the fixed straight line about which the semicircle revolves.

XVI.

the centre of a sphere is the same with that of the semicircle,

XVII.

the diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

XVIII.

cone is a solid figure described by the revolution of a right angled triangle about one of the sides containing the right angle, which side remains fixed.

XIX.

the axis of a cone is the fixed straight line about which the triangle revolves.

XX.

Book VII.

XX.

The base of a cone is the circle described by that side containing the right angle, which revolves.

XX.

A cylinder is a solid figure described by the revolution of a right angled parallelogram about one of its sides which remains fixed.

XXI.

The axis of a cylinder is the fixed straight line about which the parallelogram revolves.

XXII.

The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

XXIII.

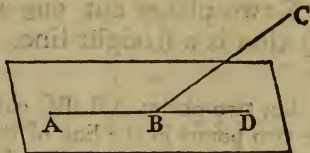
Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

PRO

PROP. I. THEOR.

ONE part of a straight line cannot be in a plane and another part above it. See N.

If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it: and since the straight line AB is in the plane, it can be produced in that plane: let it be produced to D: Then ABC and ABD are two straight lines, and they have a common segment AB, which is impossible ^b. Therefore ABC is not a straight line. Wherefore one part, &c. E. D.



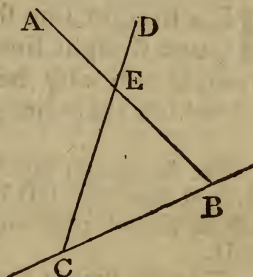
^b cor. def.
4. 1.

PROP. II. THEOR.

ANY three straight lines which meet one another, not in the same point, are in one plane.

Let the three straight lines AB, CD, CB, meet one another in the points B, C and E; AB, CD, CB are in one plane.

Let any plane pass through the straight line EB, and let the plane be turned about EB, produced, if necessary, until it pass through the point C: Then, because the points A, C are in this plane, the straight line EC is in it ^a: for the same reason, the straight line BC is in the plane; and, by the hypothesis, EB is in it: therefore the three straight lines EC, CB, BE are in one plane: and the whole of the lines DC, AB, and BC produced, are in the same plane with the parts of them



^a 5. def. 1.

EC,

Book VII. EC, EB, BC b. Therefore AB, CD, CB, are all in one plane. Wherefore, &c. Q. E. D.

b 1. 7.

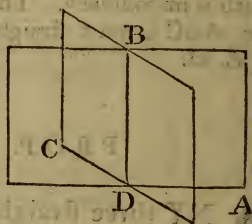
COR. It is manifest, that any two straight lines which cut one another are in one plane.

PROP. III. THEOR.

IF two planes cut one another, their common section is a straight line.

Let two planes AB, BC cut one another, and let B and D be two points in the line of their common section. From B to D draw the straight line BD; and because the points B and D are in the plane AB, the straight line BD is in that plane a: for the same reason it is in the plane CB; the straight line BD is therefore common to the planes AB and BC, or it is the common section of these planes. Therefore, &c. Q. E. D.

a 5. def 1.



PROP. IV. THEOR.

IF a straight line stand at right angles to each of two straight lines in the point of their intersection, it will also be at right angles to the plane in which these lines are.

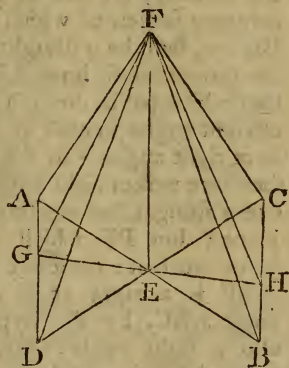
Let the straight line EF stand at right angles to each of the straight lines AB, CD in E, the point of their intersection. EF is also at right angles to the plane passing through AB, CD.

Take the straight lines AE, EB, CE, ED all equal to one another; and through E draw, in the plane in which are AB, CD

CD

Draw any straight line GEH; and join AD, CB; then, from Book VII.
 any point F in EF, draw FA, FG, FD, FC, FH, FB: and
 because the two straight lines, AE, ED are equal to the two
 EC, EC, and that they contain equal angles ^a AED, BEC, a 5. 1.
 the base AD is equal ^b to the base BC, and the angle b 4. 1.
 AEG is equal to the angle EBC: and the angle AEG is equal to
 the angle BEH ^a; therefore the triangles AEG, BEH have
 two angles of one equal to two angles of the other, each to
 each, and the sides AE, EB, adjacent to the equal angles, e-
 qual to one another; wherefore they shall have their other
 sides equal ^c: GE is therefore equal to EH, and AG to BH: c 26. 1.
 and because AE is equal to EB, and FE common and at
 right angles to them, the base AF is equal ^b to the base FB; b 4. 1.
 for the same reason, CF is equal to FD: and because AD is
 equal to BC, and AF to FB;
 and because the base DF was pro-

ved equal to the base FC; there-
 fore the angle FAD is equal ^d to
 the angle FBC. Again, it was
 proved that GA is equal to
 BH, and AF to FB; and
 the angle FAG has also been
 proved equal to the angle FBH;
 therefore the base GF is equal
 to the base FH: And because
 GE is equal to EH, as was al-
 ready proved, and EF is com-
 mon to the triangles GFE, HFE;
 and also the base GF equal to



d 8. 1.

base FH; the angle GEF is equal ^d to the angle HEF;
 consequently each of these angles is a right ^e angle. There- e 7. def. 1.
 fore FE makes right angles with GH, that is with any straight
 line drawn through E in the plane passing through AB, DC;
 therefore it makes right angles with every straight line which
 meets it in that plane. But a straight line is at right angles to
 a plane when it makes right angles with every straight line
 which meets it in that plane ^f: therefore EF is at right angles f 1. def. 7.
 to the plane in which are AB, CD. Wherefore, if a straight
 line, &c. Q. E. D.

P

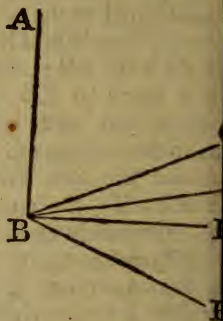
PROP.

PROP. V. THEOR.

IF three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point: these three straight lines are in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC , BD , BE , in B , the point where they meet; BC , BD , BE are in one and the same plane.

If not, let, if it be possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB , BC , the common section of which with the plane, in which BD and BE are, shall be a straight ^a line; let this be BF : therefore the three straight lines AB , BC , BF are all in one plane, viz. that which passes through AB , BC ; and because AB stands at right angles to each of the straight lines BD , BE , it is ^b so at right angles ^b to the plane passing through them; and ^c therefore makes right angles ^c with every straight line meeting it in that plane; but BF which is in that plane meets it: therefore the angle ABF is a right angle; but the angle ABC , by the hypothesis, is also a right angle; therefore the angle ABF is equal to the angle ABC , and they are both in the same plane, which is impossible: therefore the straight line BC is not above the plane in which are BD and BE : Wherefore the three straight lines BC , BD , BE are in one and the same plane. Therefore, if three straight lines, &c. Q. E. D.



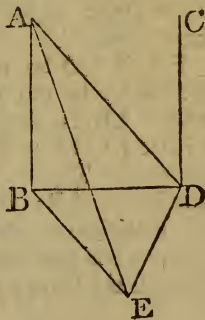
PROP. VI. THEOR.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Let the straight lines AB , CD be at right angles to the same plane; AB is parallel to CD .

Let them meet the plane in the points B , D , and draw the straight line BD , to which draw DE at right angles, in the same plane; and make DE equal to

AB , and join BE , AE , AD . Then, because AB is perpendicular to the plane, it shall make right angles with every straight line which meets it, and is in that plane: but BD , BE , which are in that plane, do each of them meet AB . Therefore each of the angles ABD , ABE is a right angle: for the same reason, each of the angles DCB , CDE is a right angle: and because AB is equal to DE , and BD common, the two sides AB , BD , are equal to the two ED , DB ; and



a 1. def. 7.

they contain right angles; therefore the base AD is equal ^b to the base BE . Again, because AB is equal to DE , and BE to BD , and the base AE common to the triangles ABE , EDA ;

the angle ABE is equal ^c to the angle EDA : but ABE is a right angle; therefore EDA is also a right angle, and ED perpendicular to DA : but it is also perpendicular to each of the

to BD , DC : Wherefore ED is at right angles to each of the three straight lines BD , DA , DC in the point in which they meet: Therefore these three straight lines are all in the same plane ^d: but AB is in the plane in which are BD , DA ,

because any three straight lines which meet one another are in the same plane ^e: Therefore AB , BD , DC are in one plane; and each of the angles ABD , BDC is a right angle; therefore AB is parallel ^f to CD . Wherefore, if two straight lines, &c.

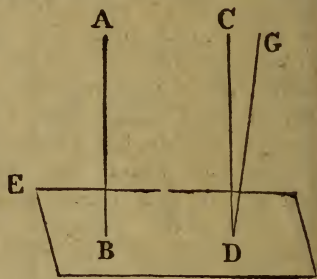
$E. D.$

PROP. VII. THEOR.

IF two straight lines be parallel, and one of them is at right angles to a plane; the other also shall be at right angles to the same plane.

Let AB , CD be two parallel straight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the same plane.

For, if CD be not perpendicular to the plane to which AB is perpendicular, let DG be perpendicular to it. Then ^a DG is parallel to AB : DG and DC therefore are both parallel to AB , and are drawn



^a 6. 7.

^b 11. Ax. 1. through the same point D , which is impossible ^b.

PROP. VIII. THEOR.

TWO straight lines which are each of them parallel to the same straight line, though not both in the same plane with it, are parallel to one another.

Let AB , CD be each of them parallel to EF , and not in the same plane with it; AB shall be parallel to CD .

In EF take any point G , from which draw, in the plane passing through EF , AB , the straight line GH at right angles to EF ; and in the plane passing through EF , CD , draw GK at right angles to the same EF . And because EF is perpendicular both to GH and GK , EF is perpendicular

^a 4. 7.

the plane HGK passing through them: and EF is paral- Book VII.

to AB; therefore

is at right angles b

the plane HGK. For

the same reason, CD

likewise at right

angles to the plane

HGK. Therefore AB,

CD are each of them

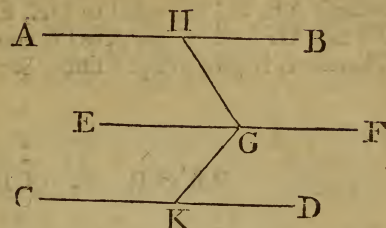
right angles to the

plane HGK. But if

two straight lines are at right angles to the same plane, they

are parallel c to one another. Therefore AB is parallel c 6. 7.

CD. Wherefore two straight lines, &c. Q. E. D.



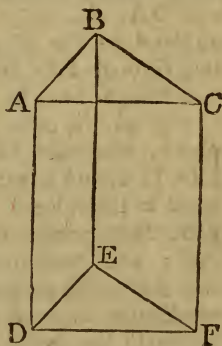
b 7. 7.

PROP. IX. THEOR.

If two straight lines meeting one another be parallel to two others that meet one another, though not in the same plane with the first two; the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC which meet one another be parallel to the two straight lines DE, EF that meet one another, and are not in the same plane with AB, BC. The angle BAC is equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AE, DF: because BA is equal and parallel to ED. therefore AD is a equal and parallel to BE. For the same reason, CF is equal and parallel to BE. Therefore AD and CF are each of them equal and parallel to BE. But straight lines that are parallel to the same straight line, though not in the same plane with it, are parallel b to one another. Therefore AD is parallel to CF; b 8. 7.



a 33. 1.

P 3

and

Book VII. and it is equal to it, and AC, DF join them towards the same parts; and therefore ^a AC is equal and parallel to DF. And because AB, BC are equal to DE, EF, and the base AC the base DF; the angle ABC is equal ^c to the angle DEF. Therefore, if two straight lines, &c. Q. E. D.

PROP. X. THEOR.

TO draw a straight line perpendicular to a plane from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a straight line perpendicular to the plane BH.

In the plane draw any straight line BC, and from the point A draw ^a AD perpendicular to BC. If then AD be also perpendicular to the plane BH, the thing required is already done; but if it be not, from

the point D draw ^b, in the plane BH, the straight line DE at right angles to BC;

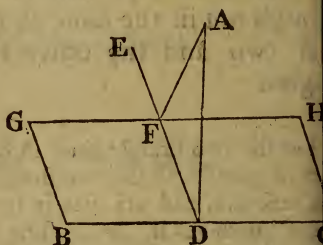
and from the point A draw AF perpendicular to DE;

and through F draw ^c GH parallel to BC: and because BC is at right angles to ED and DA, BC is at right

angles ^d to the plane passing through ED, DA. And GH is parallel to BC; but,

two straight lines be parallel, one of which is at right angles to a plane, the other shall be at right ^e angles to the same plane; wherefore GH is at right angles to the plane through

ED, DA, and is perpendicular ^f to every straight line meeting it in that plane. But AF, which is in the plane through ED, DA, meets it: Therefore GH is perpendicular to AF; and consequently AF is perpendicular to GH and AF is also perpendicular to DE: Therefore AF is perpendicular to each of the straight lines GH, DE. But a straight line stands at right angles to each of two straight lines in the point of their intersection, it shall also be at right



gles to the plane passing through them ^d. And the plane passing through ED, GH is the plane BH; therefore AF is perpendicular to the plane BH; so that, from the given point, above the plane BH, the straight line AF is drawn perpendicular to that plane. Which was to be done.

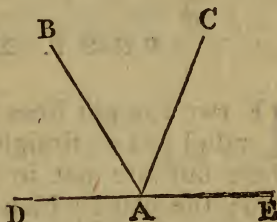
COR. If it be required from a point C in a plane to erect a perpendicular to that plane, take a point A above the plane, and draw AF perpendicular to the plane, then, if from A a line be drawn parallel to AC, it will be the perpendicular required; for being parallel to AF it will be perpendicular to the same plane to which AF is perpendicular ^e.

PROP. XI. THEOR.

FROM the same point in a given plane, there cannot be two straight lines at right angles to the plane, upon the same side of it: And there can be but one perpendicular to a plane from a point above the plane.

For, if it be possible, let the two straight lines AC, AB be at right angles to a given plane from the same point A in the plane, and upon the same side of it; and let a plane pass through BA, AC; the common section of this with the given plane is a straight line passing through A: Let DAE be their common section: Therefore the straight lines AB, AC, AE are in one plane: And because CA is at right angles to the given plane, it shall make right angles with every straight line meeting it in that plane. But AE, which is in that plane, meets CA; therefore CAE is a right angle. For the same reason BAE is a right angle. Wherefore the angle CAE is equal to the angle BAE; and they are in the same plane, which is impossible.

Also, from a point above a plane, there can be but one perpendicular to that plane; for, if there could be two, they



Book VII. would be parallel ^b to one another, which is absurd. Therefore, from the same point, &c. Q. E. D.

b 6. 7.

PROP. XII. THEOR.

PLANES to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD , EF ; these planes are parallel to one another.

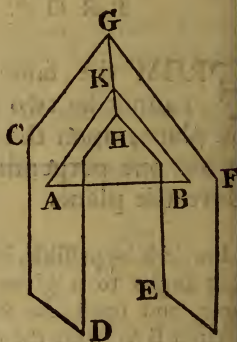
If not, they shall meet one another when produced; let them meet; their common section shall be a straight line GH , in which take any point K , and join AK , BK : Then, because AB is perpendicular to the plane EF , it is perpendicular ^a to the straight line BK which is in that plane. Therefore ABK is a right angle. For the same reason, BAK is a right angle; wherefore the two angles ABK , BAK of the triangle ABK are equal to two right angles, which is impossible ^b: Therefore the planes

a 1. def. 7.

CD , EF , though produced, do not

b 17. 1.

meet one another; that is, they are parallel ^c. Therefore planes, &c. Q. E. D.



PROP. XIII. THEOR.

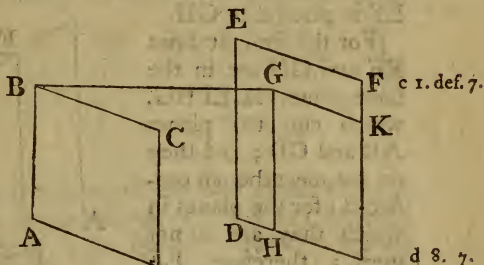
IF two straight lines meeting one another, be parallel to two straight lines which meet one another, but are not in the same plane with the first two; the plane which passes through these is parallel to the plane passing through the others.

Let

Let AB, BC, two straight lines meeting one another, be Book VII.
parallel to DE, EF that meet one another, but are not in the
the plane with AB, BC: The planes through AB, BC, and
E, EF shall not meet, though produced

From the point B draw BG perpendicular ^a to the plane ^a 10. 7.
which passes through DE, EF, and let it meet that plane in
; and through G draw GH parallel ^b to ED, and GK pa- ^b 31. 1.
ral to EF: And because BG is perpendicular to the plane
rough DE, EF, it

all make right angles
th every straight
e meeting it in that
ane ^c. But the straight
es GH, GK in that
ane meet it: There-
re each of the angles
GH, BGK is a right
gle: And because
A is parallel ^d to GH
or each of them is
ral to DE,) the



gles GBA, BGH are together equal ^e to two right angles: ^e 29. 1.
nd BGH is a right angle; therefore also GBA is a
ght angle, and GB perpendicular to BA: For the same
ason, GB is perpendicular to BC: Since therefore the
aight line GB stands at right angles to the two straight lines
A, BC, that cut one another in B; GB is perpendicular ^f ^f 4. 7.
the plane through BA, BC: And it is perpendicular to the
ne through DE, EF; therefore BG is perpendicular to
ch of the planes through AB, BC, and DE, EF: But planes
which the same straight line is perpendicular, are parallel ^g ^g 12. 7.
one another: Therefore the plane through AB, BC is pa-
ral to the plane through DE, EF. Wherefore, if two
aight lines, &c. Q. E. D.

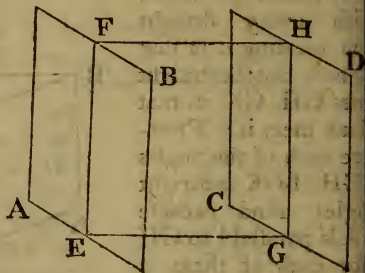
P R O P.

PROP. XIV. THEOR.

IF two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes AB , CD be cut by the plane $EFHG$, and let their common sections with it be EF , GH : EF is parallel to GH .

For the straight lines EF and GH are in the same plane, viz. $EFHG$, which cuts the planes AB and CD ; and they do not meet though produced; for the planes in which they are do not meet; therefore EF and GH are parallel^a.
Q. E. D.



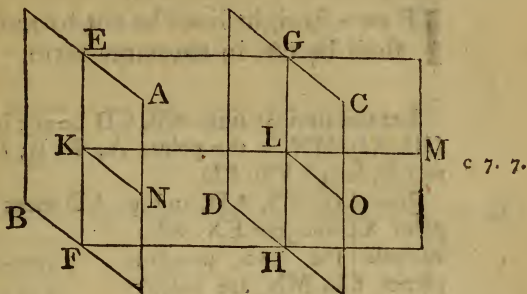
PROP. XV. THEOR.

IF two parallel planes be cut by a third plane, they have the same inclination to that plane.

Let AB and CD be two parallel planes, and EH a third plane cutting them: The planes AB and CD are equally inclined to EH .

Let the straight lines EF and GH be the common sections of the plane EH with the two planes AB and CD ; and from K any point in EF , draw in the plane EH the straight line KM at right angles to EF , and let it meet GH in L ; draw also KN at right angles to EF in the plane AB : and through the straight lines KM , KN , let a plane be made to pass cutting the plane CD in the line LO . And because EF and GH are the common sections of the plane EH with the two

two parallel planes AB and CD, EF is parallel to GH^a. Book VII.
 but EF is at right angles to the plane that passes through a 14. 7.
 KN and KM^b, be- b 4. 7.
 cause it is at right
 angles to the lines
 KM and KN:
 therefore GH is
 also at right angles
 to the same plane,
 and it is there-
 fore at right angles
 to the lines LM, LO
 which it meets in
 that plane. There-
 fore, since LM and
 LO are at right
 angles to LG, the common section of the two planes CD and



EH, the angle OLM is the inclination of the plane CD to the
 plane EH^d. For the same reason the angle MKN is the in- d 4. def. 7.
 clination of the plane AB to the plane EH. But because
 KM and LO are parallel, being the common sections of the
 parallel planes AB and CD with a third plane, the interior
 angle NKM is equal to the exterior angle OLM^e; that is, the e 29. 1.
 inclination of the plane AB to the plane EH, is equal to the
 inclination of the plane CD to the same plane EH. There-
 fore, &c. Q. E. D.

PROP.

PROP. XVI. THEOR.

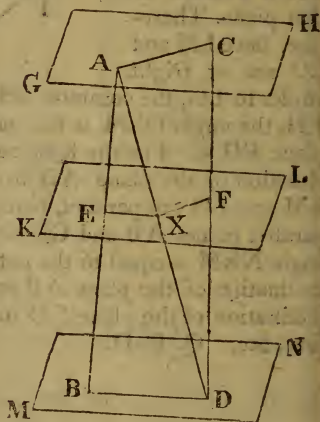
IF two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the straight lines AB, CD be cut by the parallel planes GH, KL, MN , in the points $A, E, B; C, F, D$: As AE is to EB , so is CF to FD .

Join AC, BD, AD , and let AD meet the plane KL in the point X ; and join EX, XF :

Because the two parallel planes KL, MN are cut by the plane $EBDX$, the common sections EX, BD , are parallel^a. For the same reason, because the two parallel planes GH, KL are cut by the plane $AXFC$, the common sections AC, XF are parallel:

And because EX is parallel to BD , a side of the triangle ABD , as AE to EB , so is AX to XD . Again, because XF is parallel to AC , a side of the triangle ADC , as AX to XD , so is CF to FD : And it was proved that AX is to XD , as AE to EB : Therefore^c, as AE to EB , so is CF to FD . Wherefore, if two straight lines, &c. Q. E. D.



PROP. XVII. THEOR.

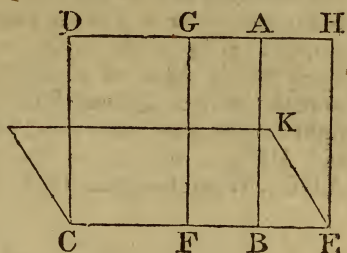
IF a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let

Let the straight line AB be at right angles to a plane CK ; every plane which passes through AB shall be at right angles to the plane CK . Book VII.

Let any plane DE pass through AB , and let CE be the common section of the planes DE, CK ; take any point F in CE , from which draw FG in the plane DE at right angles to CE : And because AB is

perpendicular to the plane CK , therefore it is also perpendicular to every straight line meeting it in that plane^a; and consequently it is perpendicular to CE : Wherefore ABF is a right angle; but GFB is likewise a right angle; therefore AB is parallel to



a 1. def. 7.

b 28. 1.

to FG . And AB is at right angles to the plane CK ; therefore FG is also at right angles to the same plane^c. But one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles to the other plane^d; and any straight line FG in the plane DE , which is at right angles to CE the common section of the planes, has been proved to be perpendicular to the other plane CK ; therefore the plane DE is at right angles to the plane CK . In like manner, it may be proved that all the planes which pass through AB are at right angles to the plane CK . Therefore, if a straight line, &c. Q. E. D.

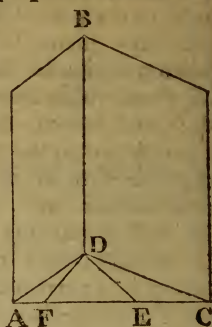
PROP. XVIII. THEOR.

If two planes cutting one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the first two; BD is perpendicular to the plane ADC .

From

Book VII. From D in the plane ADC, draw DE perpendicular to AD, and DF to DC. Because DE is perpendicular to AD the common section of the planes AB and ADC; and because the plane AB is at right angles to ADC, DE is at right angles to the plane AB^a, and therefore also to the straight line BD in that plane b. For the same reason, DF is at right angles to DB. Since BD is therefore at right angles to both the lines DE and DF, it is at right angles to the plane in which DE and DF are, that is, to the plane ADC^c. Wherefore, &c. Q. E. D.

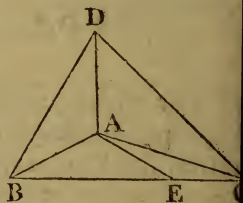


PROP. XIX. THEOR.

IF a solid angle be contained by three plane angles any two of them are greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB. Any two of them are greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. But if they are not, let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB; and at the point A in the straight line AB, make, in the plane which passes through BA, AC, the angle BAE equal to the angle DAB; and make AE equal to AD, and through E draw BEC cutting AB, AC in the points B, C, and join DB, DC. And because DA is equal to AE, and AB is common to the two triangles ABD, ABE and also the angle DAB equal to the angle EAB; there-
fo



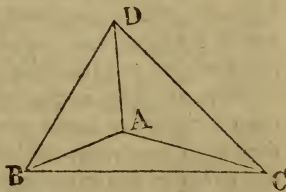
fore the base DB is equal ^b to the base BE. And be- ^{Book VII.}
 cause BD, DC are greater ^c than CB, and one of them BD ^{b 4. 1.}
 has been proved equal to BE a part of CB, therefore the o- ^{c 20. 1.}
 ther DC is greater than the remaining part EC. And be-
 cause DA is equal to AE, and AC common, but the base
 DC greater than the base EC; therefore the angle DAC is
 greater ^d than the angle EAC; and, by the construction, the ^{d 25. 1.}
 angle DAB is equal to the angle BAE; wherefore the angles
 DAB, DAC are together greater than BAE, EAC, that is,
 than the angle BAC. But BAC is not less than either of
 the angles DAB, DAC; therefore BAC, with either of
 them, is greater than the other. Wherefore, if a solid angle,
 &c. Q. E. D.

P R O P. XX. T H E O R.

EVERY solid angle is contained by plane angles
 which together are less than four right angles.

First, Let the solid angle at A be contained by three plane
 angles BAC, CAD, DAB. These three together are less
 than four right angles.

Take in each of the straight lines AB, AC, AD any
 points B, C, D, and join BC, CD, DB: then, because the so-
 lid angle at B is contained by the three plane angles CBA,
 ABD, DBC, any two of them are greater ^a than the third; ^{a 19. 1.}
 therefore the angles CBA, ABD are greater than the angle
 DBC: for the same reason, the angles BCA, ACD are great-
 er than the angle DCB; and
 the angles CDA, ADB greater
 than BDC: Wherefore the six
 angles CBA, ABD, BCA,
 ACD, CDA, ADB are great-
 er than the three angles DBC,
 BCD, CDB: but the three
 angles DBC, BCD, CDB are
 equal to two right angles ^b:



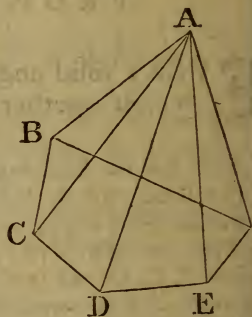
^{b 23. 1.}

Therefore the six angles CBA, ABD, BCA, ACD, CDA,
 ADB are greater than two right angles: and because the
 three

Book VII. three angles of each of the triangles ABC , ACD , ADB are equal to two right angles, therefore the nine angles of these three triangles, viz. the angles CBA , BAC , ACB , ACD , CDA , DAC , ADB , DBA , BAD are equal to six right angles: Of these, the six angles CBA , ACB , ACD , CDA , ADB , DBA are greater than two right angles; therefore the remaining three angles BAC , DAC , BAD which contain the solid angle at A , are less than four right angles.

Next, Let the solid angle at A be contained by any number of plane angles BAC , CAD , DAE , EAF , FAB ; the angles together are less than four right angles.

Let the planes in which the angles are, be cut by a plane and let the common sections of it with those planes be BC , CD , DE , EF , FB : and because the solid angle at A is contained by three plane angles CBA , ABF , FBC , of which any two are greater ^a than the third, the angles CBA , ABF are greater than the angle FBC : for the same reason, the two plane angles at each of the points C , D , E , F , viz. the angles which are at the bases of the triangles having the common vertex A , are greater than the third angle at the same point, which is one of the angles of the polygon $BCDEF$: therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon: and because all the angles of the triangles are together equal to twice as many right angles as there are triangles; that is, as there are sides in the polygon $BCDEF$; and because all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon ^c; therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the polygon as has been proved. Wherefore, the remaining angles of the triangles, viz. those at the vertex, which contain the solid angle



^a 19. 7. two are greater ^a than the third, the angles CBA , ABF are greater than the angle FBC : for the same reason, the two plane angles at each of the points C , D , E , F , viz. the angles which are at the bases of the triangles having the common vertex A , are greater than the third angle at the same point, which is one of the angles of the polygon $BCDEF$: therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon: and because all the angles of the triangles are together equal to twice as many right angles as there are triangles; that is, as there are sides in the polygon $BCDEF$; and because all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon ^c; therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the polygon as has been proved. Wherefore, the remaining angles of the triangles, viz. those at the vertex, which contain the solid angle

angle at A, are less than four right angles. Therefore every Book VII.
solid angle, &c. Q. E. D.

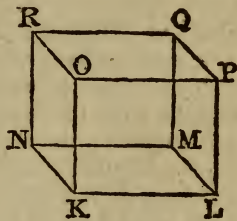
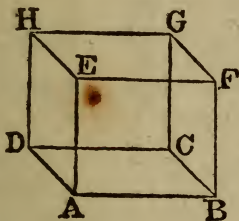
P R O P. XXI. T H E O R.

If two solids be contained by the same number of equal and similar planes, similarly situated, and the inclination of any two contiguous planes in the one solid be the same with the inclination of the two equal, and similarly situated planes in the other, the solids themselves are equal and similar.

Let AG and KQ be two solids contained by the same number of equal and similar planes, similarly situated, so that the plane AC is similar and equal to the plane KM, the plane AF to the plane KP; BG to LQ, GD to QN, DE to KO, and FH to PR. Let also the inclination of the plane AF to the plane AC be the same with that of the plane KP to the plane KM, and so of the rest; the solid KQ is equal and similar to the solid AG.

Let the solid KQ be applied to the solid AG, so that the planes KM and

AC, which are equal and similar, may coincide^a, the point N coinciding with the point D, with A, L with B, and



^a 8. Ax. 1.

on. And because the plane KM coincides with the plane AC, and, by hypothesis, the inclination of KR to KM is the same with the inclination of AH to AC, the plane KR will be upon the plane AH, and will coincide with it, because they are similar and equal^a, and because their equal sides KN and AD coincide. And in the same manner, it is shewn that the other planes of the solid KQ coincide with the other planes of the solid AG, each with each: therefore the solids KQ and AG do wholly coincide, and are equal and similar to one another. Therefore, &c. Q. E. D.

Q

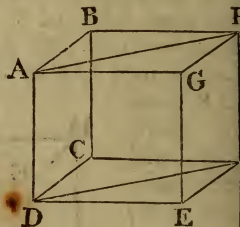
P R O P.

PROP. XXII. THEOR.

IF a solid be contained by six planes, two and two of which are parallel, the opposite planes are similar and equal parallelograms.

Let the solid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE: its opposite planes are similar and equal parallelograms.

Because the two parallel planes BG, CE, are cut by the plane AC, their common sections AB, CD, are parallel ^a. Again, because the two parallel planes BF, AE, are cut by the plane AC, their common sections AD, BC are parallel and AB is parallel to CD; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram: Join AH, DF; and because AB is parallel to DC, and BH to CF; the two straight lines AB, BH, which meet one another, are parallel to DC and CF, which meet one another; wherefore, tho' the first two are not in the same plane with the other two, they contain equal angles ^b; the angle ABH is therefore equal to the angle DCF.



And because AB, BH, are equal to DC, CF, and the angle ABH equal to the angle DCF; therefore the base AH is equal ^c to the base DF, and the triangle ABH to the triangle DCF: For the same reason, the triangle AGH is equal to the triangle DEF; and therefore the parallelogram BG is equal and similar to the parallelogram CE. In the same manner it may be proved, that the parallelogram AC is equal and similar to the parallelogram GF, and the parallelogram AE to BF. Therefore, if a solid, &c. Q. E. D.

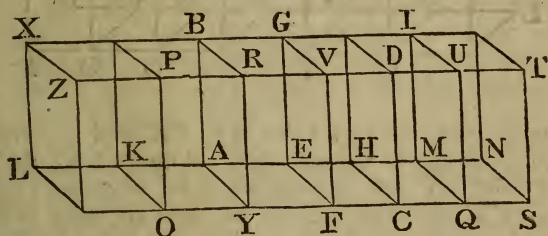
P R O

PROP. XXIII. THEOR.

If a solid parallelepiped be cut by a plane parallel to two of its opposite planes, it will be divided to two solids, which will be to one another as their bases.

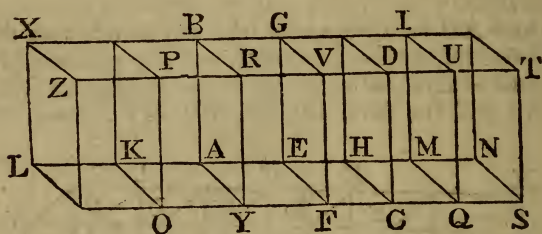
Let the solid parallelepiped $ABCD$ be cut by the plane V , which is parallel to the opposite planes AR , HD , and divides the whole into the solids $ABFV$, $EGCD$; as the base $AEFY$ to the base $EHCF$, so is the solid $ABFV$ to the solid $EGCD$.

Produce AH both ways, and take any number of straight lines HM , MN , each equal to EH , and any number AK , KL each equal to EA , and complete the parallelograms LO , KY , Q , MS , and the solids LP , KR , HU , MT : then, because



the straight lines LK , KA , AE are all equal, and also the straight lines KO , AY , EF , which make equal angles with LK , KA , AE , the parallelograms LO , KY , AF are equal and similar ^a: and likewise the parallelograms KX , ^{a 20. 6.} B ; AG ; as also ^b the parallelograms LZ , KP , AR , ^{b 22. 7.} because they are opposite planes. For the same reason, the parallelograms EC , HQ , MS , are equal ^a; and the parallelograms HG , HI , IN , as also ^b HD , MU , NT ; therefore three planes of the solid LP , are equal and similar to three planes of the solid KR , as also to three planes of the solid AV : but the three planes opposite to these three are equal and similar to them ^b in the several solids; therefore the solids LP , KR , AV are contained by equal and similar planes. And because the planes LZ , KP , AR are parallel, and are cut by the plane XV , the inclination of LZ

Book VII. to XP is equal to that of KP to PB, or of AR to BV ^c : and
^c 15. 7. the same is true of the other contiguous planes, therefore the
^d 21. 7. solids LP, KR, and AV, are equal to one another ^d. For the
 same reason, the three solids ED, HU, MT are equal to one
 another; therefore what multiple soever the base LF is
 the base AF, the same multiple is the solid LV of the solid
 AV; for the same reason, whatever multiple the base NF
 of the base HF, the same multiple is the solid NV of the
 solid ED: and if the base LF be equal to the base NF, the
 solid LV is equal ^d to the solid NV; and if the base LF be
 greater than the base NF, the solid LV is greater than
 the solid NV; and if less, less. Since then there are four



magnitudes, viz. the two bases AF, FH, and the two solids AV, ED, and of the base AF and solid AV, the base LF and solid LV are any equimultiples whatever; and of the base FH and solid ED, the base FN and solid NV are any equimultiples whatever; and it has been proved, that the base LF is greater than the base FN, the solid LV greater than the solid NV; and if equal, equal; and if less, less: Therefore ^e, as the base AF is to the base FH, so the solid AV to the solid ED. Wherefore, if a solid, &c. Q. E. D.

^f 1. 6. COR. Because the parallelogram AF is to the parallelogram FH as YF to FC ^f, therefore the solid AV is to the solid ED as YF to FC.

PROP. XXIV. THEOR.

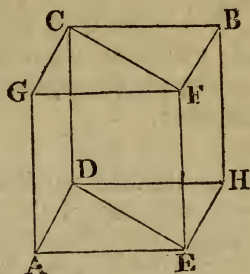
IF a solid parallelepiped be cut by a plane passing through the diagonals of two of the opposite planes, it shall be cut in two equal prisms.

Book VII.

Let AB be a solid parallelepiped, and DE, CF the diagonals of the opposite parallelograms AH, GB, viz. those which are drawn betwixt the equal angles in each: and because D, FE are each of them parallel to GA, though not in the same plane with it, CD, FE are parallel ^a; wherefore the diagonals CF, DE are in the plane which the parallels are, and are themselves parallels ^b: and the plane DEF shall cut the solid AB into two equal parts

Because the triangle CGF is equal to the triangle CBF, and the triangle DAE to DHE; and that the parallelogram CA is equal ^d and similar to the opposite one BE; and the parallelogram GE to CH: therefore the planes which contain

the prisms CAE, CBE, are equal and similar, each to each; and they are also equally inclined to one another, because the planes AC, EB are parallel, as also AF and BD, and they are cut by the plane CE ^c. Therefore the prism CAE is equal to the prism CBE ^e, and the solid AB is cut into two equal prisms by the plane CDEF. Q. E. D.



b 14. 7.

c 34. 1.

d 22. 7.

e 21. 7.

N. B. The insisting straight lines of a parallelepiped, mentioned in the following propositions, are the sides of the parallelograms betwixt the base and the plane parallel to it.

Q. 3

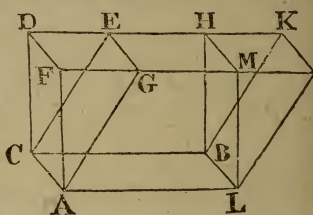
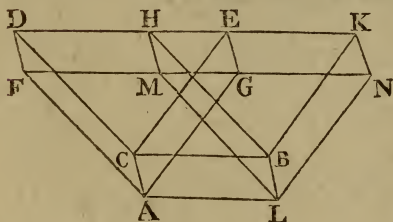
P R O P.

PROP. XXV. THEOR.

SOLID parallelepipeds upon the same base, and of the same altitude, the inscribing straight lines of which are terminated in the same straight line in the plane opposite to the base, are equal to one another.

Let the solid parallelepipeds AH, AK be upon the same base AB, and of the same altitude, and let their inscribing straight lines AF, AG, LM, LN, be terminated in the same straight line FN, and CD, CE, BH, BK be terminated in the same straight line DK; the solid AH is equal to the solid AK.

- a 34. 1. Because CH, CK are parallelograms, CB is equal ^a to each of the opposite sides DH, EK; wherefore DH is equal to EK: add, or take away the common part HE; then DE is equal to HK: Wherefore also the triangle CDE is equal ^b to the triangle BHK: and the parallelogram DG is equal ^c to the parallelogram HN. For the same reason, the triangle AFG is equal to the triangle LMN, and the parallelogram CF is equal ^d to the parallelogram BM, and CG to BN;



they are opposite. Therefore the planes which contain the prism DAG are similar and equal to those which contain the prism HLN, each to each; and the contiguous planes are also equally inclined to one another ^e, because that the parallel planes AD and LH, as also AE and LK, are cut by the same plane DN: therefore the prisms DAG, HLN are equal ^f. therefore

e 15. 7.

f 21. 7.

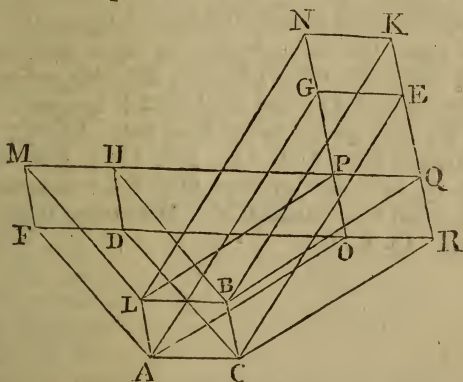
Therefore the prism LNH be taken from the solid, of which Book VII.
 the base is the parallelogram AB, and FDKN the plane opposite to the base; and if from this same solid there be taken
 the prism AGD, the remaining solid, viz. the parallelepiped
 H, is equal to the remaining parallelepiped AK. Therefore
 the solid parallelepipeds, &c. Q. E. D.

PROP. XXVI. THEOR.

SOLID parallelepipeds upon the same base, and
 of the same altitude, the insisting straight lines
 which are not terminated in the same straight
 lines in the plane opposite to the base, are equal to
 one another.

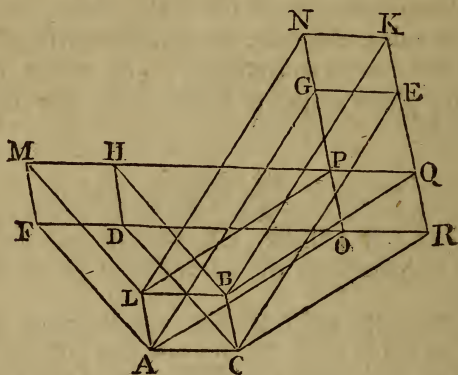
Let the parallelepipeds CM, CN, be upon the same base
 B, and of the same altitude, but their insisting straight
 lines AF, AG, LM, LN, CD, CE, BH, BK, not terminated
 in the same straight lines; the solids CM, CN are equal to
 one another.

Produce FD, MH, and NG, KE, and let them meet one
 another in the points O, P, Q, R; and join AO, LP, BQ,
 CR. Because the planes ^a LBHM and ACDF are pa- a12. def. 7.



allel, and because the plane LBHM is that in which are
 the parallels LB, MHPQ^a, and in which also is the figure
 Q 4 BLPQ;

Book VII. BLPQ; and because the plane ACDF is that in which are the parallels AC, FDOR, and in which also is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes. In like manner, because the planes ALNG and CBKE are parallel, and the plane ALNG is that in which are the parallels AL, OPGN, and in which also is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, and in which also is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes. But the planes ACBL, ORQP are also parallel; therefore the solid CP is a parallelepiped. Now the solid para



b 25. 7. lelepiped CM is equal^b to the solid parallelepiped CP; because they are upon the same base, and their insisting straight lines AF, AO, CD, CR; LM, LP, BH, BQ are in the same straight lines FR, MQ: and the solid CP is equal^b to the solid CN; for they are upon the same base ACBL, and the insisting straight lines AO, AG, LP, LN; CR, CE, BQ, BK are in the same straight lines ON, RK: therefore the solid CM is equal to the solid CN. Wherefore solid parallelepipeds, &c. Q. E. D.

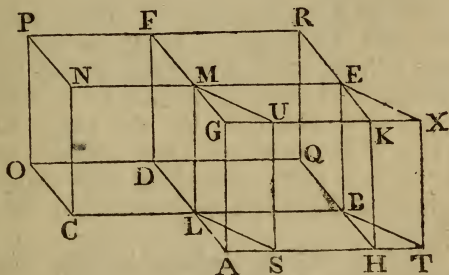
PROP. XXVII. THEOR.

SOLID parallelepipeds which are upon equal bases, and of the same altitude, are equal to one another.

Let the solid parallelepipeds, AE , CF , be upon equal bases AB , CD , and be of the same altitude: the solid AE is equal to the solid CF .

Case 1. Let the insisting straight lines be at right angles to the bases AB , CD , and let the bases be placed in the same plane, and so as that the sides CL , LB be in a straight line; therefore the straight line LM , which is at right angles to the plane in which the bases are, in the point L , is common ^{a 11. 7.} to the two solids AE , CF ; let the other insisting lines of the solids be AG , HK , BE ; DF , OP , CN : and first, let the angle ALB be equal to the angle CLD ; then AL , LD are in a straight line ^{b 14. 1.}. Produce OD , HB , and let them meet in Q , and complete the solid parallelepiped LR , the base of which is the parallelogram LQ , and of which LM is one of its insisting straight lines: therefore, because the parallelogram AB is equal to CD , as the base AB is to the base LQ , so is ^{c 7. 5.} the base CD to the same LQ : and because the solid parallelepiped AR is cut by

the plane $LMEB$, which is parallel to the opposite planes AK , DR ; as the base AB is to the base LQ , so is ^{d 23. 7.} the solid AE to the solid LR : for the same reason, because the solid parallelepiped CR is cut by the plane $LMFD$, which is parallel to the opposite planes CP , BR ; as the base CD to the base LQ ; so is the solid CF to the



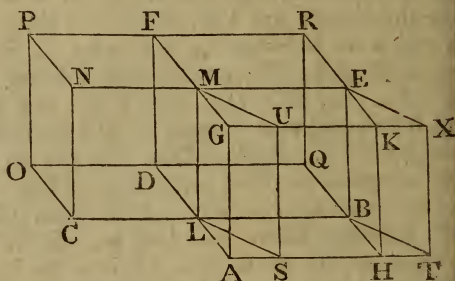
Book VII. the solid LR : but as the base AB to the base LQ, so the base CD to the base LQ, as has been proved : therefore as the solid AE to the solid LR, so is the solid CF to the solid LR ; and therefore the solid AE is equal^e to the solid CF.

e 9. 5.

But let the solid parallelepipeds SE, CF be upon equal bases SB, CD, and be of the same altitude, and let their insisting straight lines be at right angles to the bases ; and place the bases SB, CD in the same plane, so that CL, LB be in a straight line ; and let the angles SLB, CLD be unequal ; the solid SE is also in this case equal to the solid CF. Produce DL, TS until they meet in A, and from B draw BH parallel to DA ; and let HB, OD produced meet in Q, and complete the solids AE, LR : therefore the solid AE, of which the base is the parallelogram LE, and AK the plane opposite

f 25. 7.

to it, is equal^f to the solid SE, of which the base is LE, and SX the plane opposite ; for they are upon the same base LE, and of the same altitude, and their insisting straight lines, viz. LA, LS, BH, BT ; MG, MU, EK, EX are



in the same straight lines AT, GX : and because the parallelogram AB is equal^g to SB, for they are upon the same base LB, and between the same parallels LB, AT ; and because the base SB is equal to the base CD ; therefore the base AB is equal to the base CD ; but the angle ALB is equal to the angle CLD : therefore, by the first case, the solid AE is equal to the solid CF ; but the solid AE is equal to the solid SE, as was demonstrated ; therefore the solid SE is equal to the solid CF.

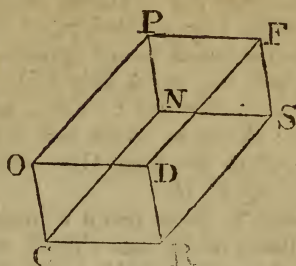
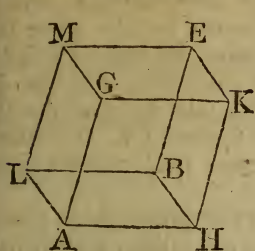
g 35. 1.

Case 2. If the insisting straight lines AG, HK, BE, LM ; CN, RS, DF, OP, be not at right angles to the bases AB, CD ; in this case likewise the solid AE is equal to the solid CF. Because solid parallelepipeds on the

h 26. 7

same base, and of the same altitude, are equal^h, if two solid

and parallelepipeds be constituted on the bases AB and CD Book VII.



of the same altitude with the solids AE and CF, and with their insisting lines perpendicular to their bases, they will be equal to the solids AE and CF; and, by the first case of this proposition, they will be equal to one another; wherefore, the solids AE and CF are also equal. Wherefore, solid parallelepipeds, &c. Q. E. D.

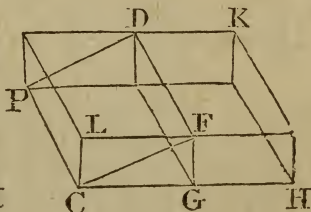
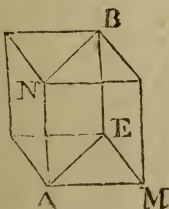
PROP. XXVIII. THEOR.

SOLID parallelepipeds which have the same altitude, are to one another as their bases.

Let AB, CD be solid parallepipeds of the same altitude : they are to one another as their bases ; that is, as the base AE to the base CF, so is the solid AB to the solid CD.

To the straight line FG apply the parallelogram FH equal

to AE, so that the angle FGH be equal to the angle LCG ; and complete the solid parallelepiped GK upon the base



a cor. 45. r.

FH, one of whose insisting lines is FD, whereby the solids CD, GK must be of the same altitude. Therefore the solid

Book VII. solid AB is equal ^b to the solid GK, because they are upon equal bases AE, FH, and are of the same altitude: and because the solid parallelepiped CK is cut by the plane DG which is parallel to its opposite planes, the base HF is ^c to the base FC, as the solid HD to the solid DC: But the base HF is equal to the base AE, and the solid GK to the solid AB: therefore, as the base AE to the base CF, so is the solid AB to the solid CD. Wherefore solid parallelepipeds, &c. Q. E. D.

COR. 1. From this it is manifest, that prisms upon triangular bases, of the same altitude, are to one another as their bases. Let the prisms BNM, DPG the bases of which are the triangles AEM, CFG, have the same altitude; complete the parallelograms AE, CF, and the solid parallelepipeds AB, CD, in the first of which let AN, and in the other let CP be one of the insisting lines. And because the solid parallelepipeds AB, CD have the same altitude, they are to one another as the base AE is to the base CF; wherefore the prisms, which are their halves ^d are to one another, as the base AE to the base CF; that is, as the triangle AEM to the triangle CFG.

COR. 2. Also a prism and a parallelepiped, which have the same altitude, are to one another as their bases; that is, the prism BNM is to the parallelepiped CD as the triangle AEM to the parallelogram LG. For by the last Cor. the prism BNM is to the prism DPG as the triangle AEM to the triangle CGF, and therefore the prism BNM is to twice the prism DPG as the triangle AEM to twice the triangle CGF ^e; that is, the prism BNM is to the parallelepiped CD as the triangle AEM to the parallelogram LG.

PROP. XXIX. THEOR.

SOLID parallelepipeds are to one another in the ratio that is compounded of the ratios of their bases, and of their altitudes.

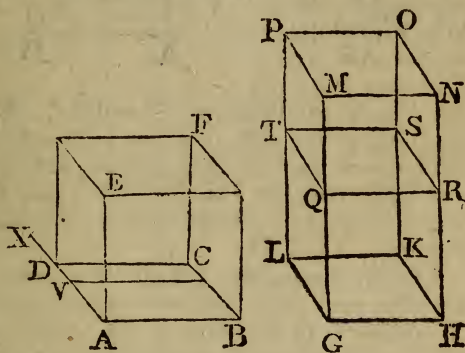
Let AF and GO be two solid parallelepipeds, of which the bases are the parallelograms AC and GK, and the altitudes, the perpendiculars let fall on the planes of these bases from any point in the opposite planes EF and MO; the solid AF is to the solid GO in a ratio compounded of the ratios of the base AC to the base GK, and of the perpendicular AF to the perpendicular on GK.

Case 1. When the insisting lines are perpendicular to the bases AC and GK, or when the solids are upright.

In GM, one of the insisting lines of the solid GO, take Q equal to AE, one of the insisting lines of the solid AF, and through Q let a plane pass parallel to the plane GK, meeting the other insisting lines of the solid GO in the points R, S and

It is evident that GS is a solid parallelepiped^a, and that it has the same altitude with AF, viz. GQ or AE. Now the solid AF is to the solid GO in a ratio compounded of the ratios of the solid AF to

the solid GS^b, and of the solid GS to the solid GO; but the^b 10. def. 5. ratio of the solid AF to the solid GS, is the same with that of the base of AC to the base GK^c, because their altitudes^c 28. 7. E and GQ are equal; and the ratio of the solid GS to the solid GO, is the same with that of GQ to GM^d; ^d 23. 7. therefore, the ratio which is compounded of the ratios of the solid AF to the solid GS, and of the solid GS to the



a 22. 7.

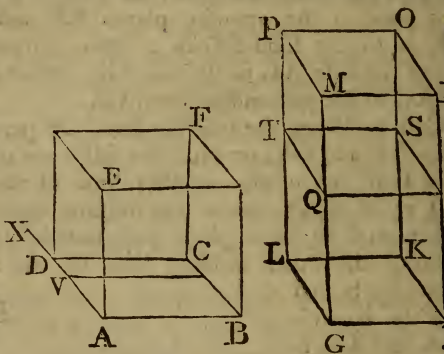
Book VII.

e E. 5.

the solid GO, is the same with the ratio which is compounded of the ratios of the base AC to the base GK and of the altitude AE to the altitude GM. But the ratio of the solid AF to the solid GO, is that which is compounded of the ratios of AF to GS, and of GS to GO; therefore, the ratio of the solid AF to the solid GO is compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM.

Case 2. When the insisting lines are not perpendicular to the bases.

Let the parallelograms AC and GK be the bases as before, and let AE and GM be the altitudes of two parallelepipeds Y and Z on these bases. Then, if the upright parallelepipeds AF and GO be constituted on the bases AC and GK, with the altitudes AE and GM, they will be



d 27. 7.

equal to the parallelepipeds Y and Z. Now, the solids AF and GO, by the first case, are in the ratio compounded of the ratios of the bases AC and GK, and of the altitudes AE and GM; therefore also, the solids Y and Z have to one another a ratio that is compounded of these same ratios. Therefore, &c. Q. E. D.

COR. 1. Hence, two straight lines may be found having the same ratio with the two parallelepipeds AF and GO. To AB, one of the sides of the parallelogram AC, apply the parallelogram BV equal to GK, having an angle equal to the angle BAD; and as AE to GM, so let AV be to AX; then AD is to AX as the solid AF to the solid GO. For the ratio of AD to AX is compounded of the ratios of AD to AV, and of AV to AX; but the ratio of AD to AV is the same with that of the parallelogram AC to the parallelogram BV or GK; and the ratio of AV to AX is the same with that of AE to GM; therefore, the ratio of AD to AX is compounded of the ratios of AC to GK, and of AE to GM.

e 44. 1.

f 12. 6.

g 10. def. 5

h 1. 6.

k E. 5.

But the ratio of the solid AF to the solid GO is compounded of the same ratios; therefore, as AD to AX, so is the solid AF to the solid GO. Book VII.

COR. 2. Hence all prisms are to one another in the ratio compounded of the ratios of their bases, and of their altitudes. For every prism is equal to a parallelepiped of the same altitude with it, and of an equal base l.

l 28. cor. 2.
7.

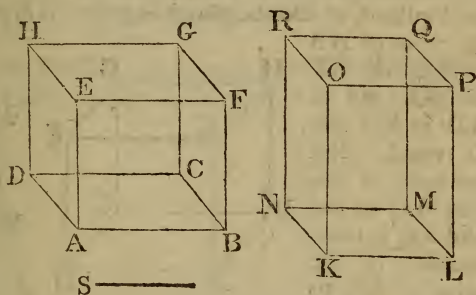
PROP. XXX. THEOR.

SOLID parallelepipeds which have their bases and altitudes reciprocally proportional, are equal; and parallelepipeds which are equal, have their bases and altitudes reciprocally proportional.

Let AG and KQ be two solid parallelepipeds, of which the bases are AC and KM, and the altitudes AE and KO, and let AC be to KM as KO to AE, the solids AG and KQ are equal.

As the base AC to the base KM, so let the straight line KO be to the straight line S. Then, since AC is to KM as KO to S, and

also by hypothesis, AC to KM as KO to AE, KO has the same ratio to S that it has to AE^a; wherefore AE is equal to S^b. But the solid



a 11. 5.

b 9. 5.

AG is to the solid KQ in the ratio compounded of the ratios of AE to KO, and of AC to KM^c; that is, in the ratio compounded of the ratios of AE to KO, and of KO to S. And the ratio of AE to S is also compounded of the same ratios^d; therefore, the solid AG has to the solid KQ the same ratio that AE has to S. But AE was proved to be equal to S, therefore AG is equal to KQ.

c 29. 7.
d 10. def. 5.

Again,

Book VII. Again, if the solids AG and KQ be equal, the base AC to the base KM as the altitude KO to the altitude AE . Take S , so that AC may be to KM as KO to S , and it will shewn, as was done above, that the solid AG is to the solid KQ as AE to S ; now, the solid AG is, by hypothesis, equal to the solid KQ ; therefore, AE is equal to S ; but, by construction, AC is to KM as KO is to S ; therefore, AC to KM as KO to AE . Therefore, &c. $Q. E. D.$

COR. In the same manner, may it be demonstrated, that equal prisms have their bases and altitudes reciprocally proportional, and conversely.

PROP. XXXI. THEOR.

SIMILAR solid parallelepipeds are to one another in the triplicate ratio of their homologous sides.

Let AG , KQ be two similar parallelepipeds, of which AE and KL are two homologous sides; the ratio of the solid AG to the solid KQ is triplicate of the ratio of AB to KL .

Because the solids are similar, the parallelograms AF , KN are similar ^a, as also the parallelograms AH , KR ; therefore the ratios of AB

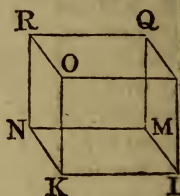
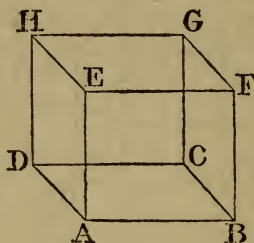
to KL , of AE to KO , and of AD to KN , are

all equal. But

the ratio of the solid AG to the solid KQ is compounded of the ratios of AC to KM , and of AE

to KO . Now, the ratio of AC to KM , because they are equiangular parallelograms, is compounded ^c of the ratios of AB to KL , and of AD to KN . Wherefore, the ratio of AG to KQ is compounded of the three ratios of AB to KL , AD to KN , and AE to KO ; and these three ratios have already been proved to be equal; therefore, the ratio that

compounde



compounded of them, viz. the ratio of the solid AG to the solid KQ, is triplicate of any of them^d; it is therefore triplicate of the ratio of AB to KL. Therefore, similar solid parallelepipeds, &c Q. E. D.

COR. 1. If as AB to KL, so KL to m , and as KL to m , so m to n , then AB is to n as the solid AG to the solid KQ. For the ratio of AB to n is triplicate of the ratio of AB to KL^e, and is therefore equal to that of the solid AG to the solid KQ.

e cor. 12.
def. 5.

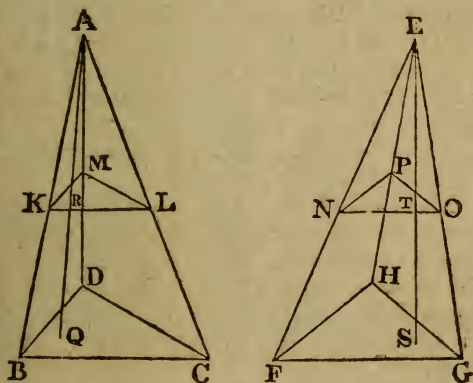
COR. 2. As cubes are similar solids, therefore the cube on AB is to the cube on KL in the triplicate ratio of AB to KL, that is in the same ratio with the solid AG to the solid KQ. Similar solid parallelepipeds are therefore to one another as the cubes on their homologous sides.

COR. 3. In the same manner it is proved, that similar prisms are to one another in the triplicate ratio, or in the ratio of the cubes, of their homologous sides.

P R O P. XXXII. T H E O R.

IF two triangular pyramids which have equal bases and altitudes be cut by planes that are parallel to the bases, and at equal distances from them, the sections are equal to one another.

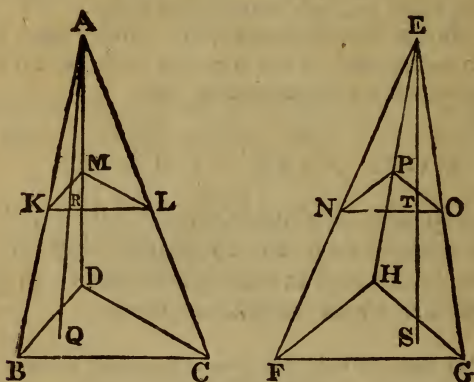
Let ABCD and EFGH be two pyramids, having equal bases BDC and FGH, and equal altitudes, viz. the perpendi-



Let ABCD and EFGH be two pyramids, having equal bases BDC and FGH, and equal altitudes, viz. the perpendi-
 culars AQ and ES drawn from A and E upon the planes BDC

Book VII. BDC and FGH ; and let them be cut by planes parallel BDC and FGH , and at equal altitudes QR and ST above those planes, and let the sections be the triangles KLM and NOP ; KLM and NOP are equal to one another.

Because the plane ABD cuts the parallel planes BDC and KLM , the common sections BD and KM are parallel ^a. For the same reason, DC and ML are parallel. Since therefore KM and ML are parallel to BD and DC , each to each, though in the same plane with them, the angle KML is equal to the angle BDC ^b. In like manner the other angles of these triangles are proved to be equal; therefore, the triangles are equiangular, and consequently similar; and the same is true of the triangles NOP , FGH .



Now, since the straight lines ARQ , AKB meet the parallel planes BDC and KLM , they are cut by them proportionally ^c, or $QR : RA :: BK : KA$; and $AQ : AR :: AB : AK$ ^d, for the same reason, $ES : ET :: EF : EN$; therefore $AB : AK :: EF : EN$, because AQ is equal to ES , and AR to ET . Again, because the triangles ABC , AKL are similar, $AB : AK :: BC : KL$; and for the same reason $EF : EN :: FG : NO$; therefore,

$BC : KL :: FG : NO$. And, when four straight lines are proportionals, the similar figures described on them are also proportionals ^e; therefore, the triangle BCD is to the triangle KLM as the triangle FGH to the triangle NOP ; but the triangles BDC , FGH are equal; therefore, the triangle KLM is also equal to the triangle NOP ^f. Therefore &c. Q. E. D.

COR. I. Because it has been shewn that the triangle KLM is similar to the base BCD , therefore, any section of a triangular pyramid parallel to the base, is a triangle similar to the

the base. And in the same manner it is shewn, that the sections parallel to the base of a polygonal pyramid are similar to the base.

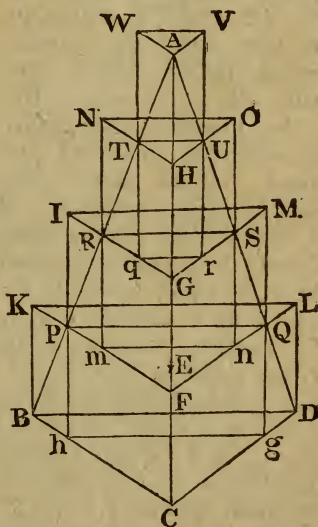
COR. 2. Hence also, the sections parallel to the bases of two polygonal pyramids, and at equal distances from the bases, are equal to one another.

PROP. XXXIII. THEOR.

A Series of prisms of the same altitude may be circumscribed about any pyramid, such that the sum of the prisms shall exceed the pyramid by a solid less than any given solid.

Let ABCD be a pyramid, and Z † a given solid; a series of prisms having all the same altitude, may be circumscribed about the pyramid ABCD, so that their sum shall exceed ABCD by a solid less than Z .

Let Z be equal to a prism standing on the same base with the pyramid, viz. the triangle BCD, and having for its altitude the perpendicular drawn from a certain point E in the line AC upon the plane BCD. It is evident, that CE multiplied by a certain number m will be greater than AC; divide A into as many equal parts as there are units in m , and let these be CF, FG, GH, HA, each of which will be less than CE. Through each of the points F, G, H let planes be made to pass parallel to the plane BCD making with the sides of the pyramid the sections FPQ, GRS, HTU, which will be all similar to one another, and to the base BCD^a. From the point B draw in the plane of the triangle ABC the straight line BK parallel to



a I. cor. 32,
7.

R 2

CF

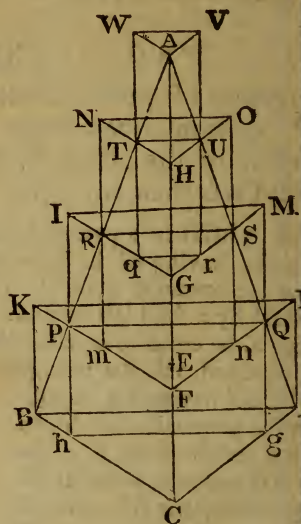
† The solid Z is not represented in the figure of this or the following Proposition.

Book VII. CF meeting FP produced in K. In like manner, from I draw DL parallel to CF, meeting FQ in L: Join KL, and
 bix. def. 7. it is plain, that the solid KBCDLF is a prism^b. By the same construction, let the prisms PM, RO, TV be described. A

dix. cor. 28

7.

so, let the straight line IP, which is in the plane of the triangle ABC be produced till it meet BC in h; and let MQ be produced till it meet DC in g: Join hg; then hCgQFP is a prism, and is equal to the prism PM^d. In the same manner is described the prism mS equal to the prism RO, and the prism qU equal to the prism TV. The sum, therefore, of all the inscribed prisms hQ, mS, and qU is equal to the sum of the prisms PM, RO and TV, that is, to the sum of all the circumscribed prisms except the prism BL; wherefore, BL is the excess of the prisms circumscribed about the pyramid ABCD above the prisms inscribed within it. But the prism BL is less than



the prism which has the triangle BCD for its base, and for its altitude the perpendicular from E upon the plane BCD; and the prism which has BCD for its base, and the perpendicular from E for its altitude, is by hypothesis equal to the given solid Z; therefore, the excess of the circumscribed, above the inscribed prisms is less than the given solid Z. But the excess of the circumscribed prisms above the inscribed is greater than their excess above the pyramid ABCD, because ABCD is greater than the sum of the inscribed prisms. Much more, therefore, is the excess of the circumscribed prisms above the pyramid, less than the solid Z. A series of prisms of the same altitude has therefore been circumscribed about the pyramid ABCD exceeding it by a solid less than the given solid Q. E. D.

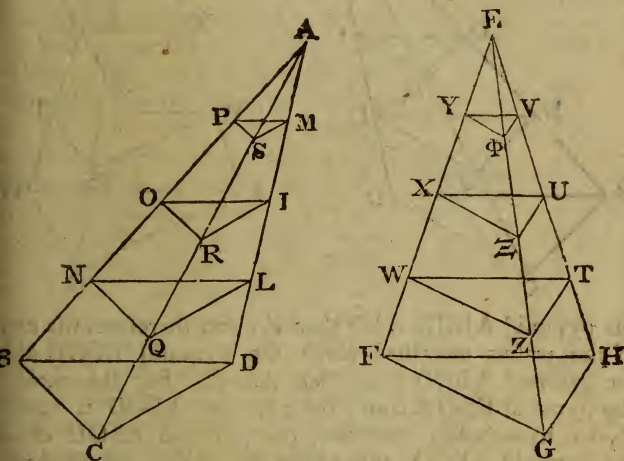
PRO

PROP. XXXIV. THEOR.

PYRAMIDS that have equal bases and altitudes are equal to one another.

Let $ABCD$, $EFGH$ be two pyramids that have equal bases BCD , FGH , and also equal altitudes, viz. the perpendiculars drawn from the vertices A and E upon the planes BCD , FGH : The pyramid $ABCD$ is equal to the pyramid $EFGH$.

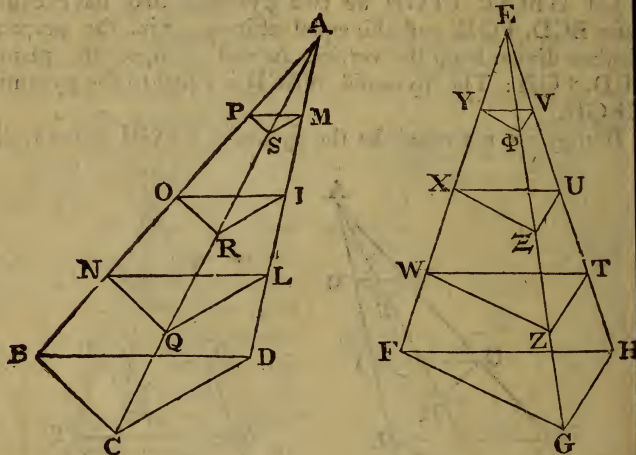
If they are not equal, let the pyramid $EFGH$ exceed the



pyramid $ABCD$ by the solid Z . Then, a series of prisms of the same altitude may be described about the pyramid $ABCD$ that shall exceed it, by a solid less than Z^a ; let these be the prisms that have for their bases the triangles BCD , NQL , ORI , PSM . Divide EH into the same number of equal parts into which AD is divided, viz. HT , TU , UV , VE , and through the points T , U and V , let the sections TZW , $U\Xi X$, $V\Phi Y$ be made parallel to the base FGH . The section NQL is equal to the section WZT^b ; as also ORI to $X\Xi U$, and PSM to $Y\Phi V$; ^{a 33. 7.} ^{b 32. 7.} and therefore, also the prisms that stand upon the equal sections

Book VII.
c. I. cor. 28.
7.

are equal c , that is, the prism which stands on the base BCD and which is between the planes BCD and NQL is equal to prism which stands on the base FGH , and which is between planes FGH and WZT ; and so of the rest, because they have the same altitude; wherefore, the sum of all the prisms described about the pyramid $ABCD$ is equal to the sum of those described about the pyramid $EFGH$. But the excess of the prisms described about the pyramid $ABCD$ above



the pyramid $ABCD$ is less than Z ; and therefore, the excess of the prisms described about the pyramid $EFGH$ above the pyramid $ABCD$ is also less than Z . But the excess of the pyramid $EFGH$ above the pyramid $ABCD$ is equal to Z , by hypothesis; therefore, the pyramid $EFGH$ exceeds the pyramid $ABCD$, more than the prisms described about $EFGH$ exceed the same pyramid $ABCD$. The pyramid $EFGH$ is therefore greater than the sum of the prisms described about it, which is impossible. The pyramids $ABCD$ and $EFGH$, therefore, are not unequal, that is, they are equal one another. Therefore, pyramids, &c. Q. E. D.

PROP. XXXV. THEOR.

EVERY prism having a triangular base may be divided into three pyramids that have triangular bases, and that are equal to one another.

Let there be a prism of which the base is the triangle ABC , and let DEF be the triangle opposite to the base: The prism $ABCDEF$ may be divided into three equal pyramids having triangular bases.

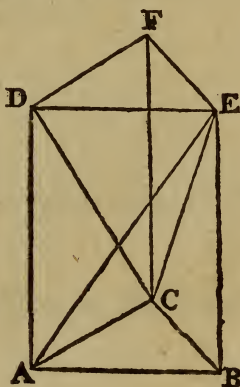
Join AE , EC , CD ; and because $ABED$ is a parallelogram, of which AE is the diameter, the triangle ADE is equal ^a to the triangle ABE : therefore the pyramid of which the base is the triangle ADE , and vertex the point C , is equal ^b to the pyramid, of which the base is the triangle ABE , and vertex the point C . But the pyramid of which the base is the triangle ABE , and vertex the point C , that is, the pyramid $ABCE$ is equal to the pyramid $DEFC$, for they have equal bases, viz. the triangles ABC , DFE , and the same altitude, viz. the altitude of the prism $ABCDEF$. Therefore, the three pyramids $ADEC$, $ABEC$, $DFEC$ are equal to one another. But the pyramids $ADEC$, $ABEC$, $DFEC$ make up the whole prism $ABCDEF$; therefore, the prism $ABCDEF$ is divided into three equal pyramids. Wherefore, &c. Q. E. D.

COR. 1. From this it is manifest, that every pyramid is the third part of a prism which has the same base, and the same altitude with it; for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

COR. 2. Pyramids of equal altitudes are to one another as their bases; because the prisms upon the same bases, and of the same altitude, are ^c to one another as their bases.

c 1. cor. 28.

7.





E L E M E N T S

OF

G E O M E T R Y.

B O O K VIII.

D E F I N I T I O N S.

I.

AN arch of a circle is any part of the circumference.

Book VIII.

II.

the perimeter of any figure is the length of the line or lines by which it is bounded.

A X I O M.

the least line that can be drawn between two points, is a straight line; and if two figures have the same straight line for their base, that which is contained within the other, if its bounding line or lines be not any where convex toward the base, has the least perimeter.

COR. 1. Hence, the perimeter of any polygon inscribed in a circle is less than the circumference of the circle.

COR. 2. If from a point two straight lines be drawn touching a circle, these two lines are together greater than the arch

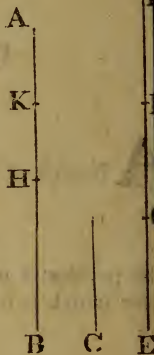
Book VIII. arch intercepted between them; and hence, the perimeter of any polygon described about a circle is greater than the circumference of the circle.

PROP. I. THEOR.

IF from the greater of two unequal magnitudes there be taken away its half, and from the remainder its half; and so on: There shall at length remain a magnitude less than the least of the proposed magnitudes.

Let AB and C be two unequal magnitudes, of which A is the greater. If from AB there be taken away its half, and from the remainder its half, and so on; there shall at length remain a magnitude less than C .

For C may be multiplied so as at length to become greater than AB . Let it be so multiplied, and let DE its multiple be greater than AB , and let DE be divided into DF, FG, GE , each equal to C . From AB take BH equal to its half, and from the remainder, AH take HK equal to its half, and so on, until there be as many divisions in AB as there are in DE : And let the divisions in AB be AK, KH, HB ; and the divisions in ED be DF, FG, GE . And because DE is greater than AB , and that EG taken from DE is not greater than its half, but BH taken from AB is equal to its half; therefore the remainder GD is greater than the remainder HA . Again, because GF is greater than HA , and that GF is not greater than the half of GD , but HK is equal to the half of HA ; therefore, the remainder FD is greater than the remainder AK . And FD is equal to C , therefore C is greater than AK ; that is, AK less than C . Q. E. D.



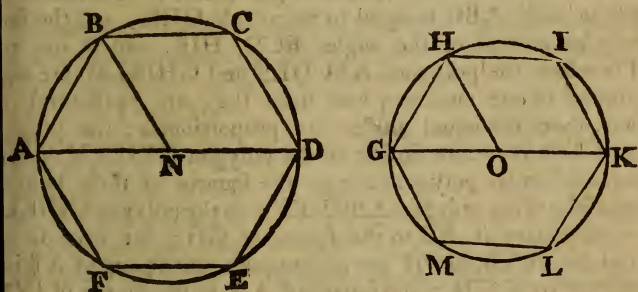
PROP.

PROP. II. THEOR.

EQUILATERAL polygons of the same number of sides inscribed in circles, are similar, and are to one another as the squares of the diameters of the circles.

Let $ABCDEF$ and $GHIKLM$ be two equilateral polygons of the same number of sides inscribed in the circles ABD , and GHK ; $ABCDEF$ and $GHIKLM$ are similar, and are to one another as the squares of the diameters of the circles ABD , GHK .

Find N and O the centres of the circles; join AN and BN , as also GO and HO , and produce AN and GO till they meet the circumspheres in D and K .

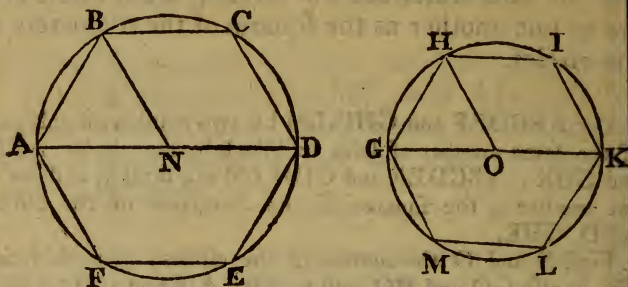


Because the straight lines AB, BC, CD, DE, EF, FA , are all equal, the arches AB, BC, CD, DE, EF, FA are also equal ^a. For the same reason, the arches GH, HI, IK, KL, LM, MG are all equal, and they are equal in number to the others; therefore, whatever part the arch AB is of the whole circumference ABD , the same is the arch GH of the circumference GHK . But the angle ANB is the same part of four right angles, that the arch AB is of the circumference ABD ^b; and the angle GOH is the same part of four right angles that the arch GH is of the circumference GHK ^b, therefore the angles ANB, GOH are each of them the same part

^a 28. 3.

^b 27. 3.

Book VIII. part of four right angles, and therefore they are equal to one another. The isosceles triangles ANB , GOH therefore equiangular, and the angle ABN equal to angle GHO ; in the same manner, by joining NC , OK it may be proved that the angles NBC , OHI are equal



to one another, and to the angle ABN . Therefore the whole angle ABC is equal to the whole GHK ; and the same may be proved of the angles BCD , HIK , and of the rest. Therefore, the polygons $ABCDEF$ and $GHIKLM$ are equiangular to one another; and since they are equilateral, their sides about the equal angles are proportionals; the polygons $ABCDEF$ and $GHIKLM$ are therefore similar. *d. 1. def. 6.* $ABCDEF$ is therefore similar to the polygon $GHIKLM$. And because similar polygons are as the squares of their homologous sides, the polygon $ABCDEF$ is to the polygon $GHIKLM$ as the square of AB to the square of GH ; but because the triangles ANB , GOH are equiangular, the square of AB is the square of GH as the square of AN to the square of GO or as four times the square of AN to four times the square of GO , that is, as the square of AD to the square of GK . *g. 15. 5. h. 7. cor. 8.* Therefore also, the polygon $ABCDEF$ is to the polygon $GHIKLM$ as the square of AD to the square of GK ; and they have also been shewn to be similar. Therefore, &c. *Q. E. D.*

COR. Every equilateral polygon inscribed in a circle is also equiangular. For the isosceles triangles, which have their common vertex in the centre, are all equal and similar; therefore the angles at their bases are all equal, and the angles of the polygon are therefore also equal.

PROP. III. PROB.

THE side of any equilateral polygon inscribed in a circle being given, to find the side of a polygon of the same number of sides described about the circle.

Let ABCDEF be an equilateral polygon inscribed in the circle ABD; it is required to find the side of an equilateral polygon of the same number of sides described about the circle.

Find G the centre of the circle; join GA, GB, bisect the chord AB in H; and through H draw KHL touching the circle in H, and meeting GA and GB produced in K and L; KL is the side of the polygon required.

Produce GF to N, so that GN may be equal to GL; join LN, and from G draw GM at right angles to KN, join also KM.

Because the circumference AB is bisected in H, the angle AGH is equal to the angle BGH^a; and because KL touches a 27. 35

circle in H, the angles

AKG, KHG are right

angles^b; therefore, there

are two angles of the tri-

angle HGK, equal to two

angles of the triangle

HL, each to each. But

the side GH is common to

these triangles; therefore

they are equal^c, and GL

is equal to GK. Again, in

triangles KGL, KGN,

because GN is equal to

GL, and GK common,

also the angle LGK

is equal to the angle KGN; therefore the base KL is equal

to the base KN^d. But because the triangle KGN is isos-

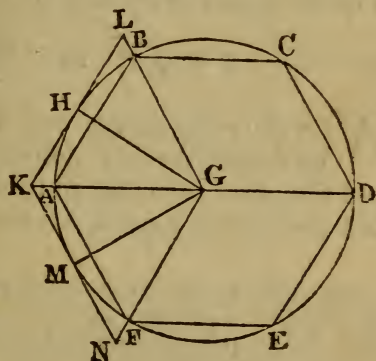
celes, the angle GKN is equal to the angle GNK, and

the angles GMK, GMN are both right angles by con-

struction; wherefore, the triangles GMK, GMN have two

sides of the one equal to two sides of the other, and they

have



b 16. 34

c 26. 23

d 4. 14

Book VIII. have also the side GM common, therefore they are equal ^c; and the side KM is equal to the side MN , so that KN is bisected in M . But KN is equal to KL , and therefore their halves KM and KH are also equal. Wherefore, in the triangles GKH , GKM , the two sides GK and KH are equal to the two GK and KM , each to each; and the angles GKH , GKM are also equal, therefore GM is equal to GH ^d; wherefore, the point M is in the circumference of the circle; and because KMG is a right angle, KM touches the circle. And in the same manner, by joining the centre and the other angular points of the inscribed polygon, an equilateral polygon may be described about the circle, the sides of which will each be equal to KL , and will be equal in number to the sides of the inscribed polygon. Therefore, KL is the side of an equilateral polygon described about the circle of the same number of sides with the inscribed polygon $ABCDEF$; which was to be found.

COR. Because GL , GK , GN , and the other straight lines drawn from the centre G to the angular points of the polygon described about the circle ABD are all equal; if a circle be described from the centre G , with the distance GK , the polygon will be inscribed in that circle; and therefore, it is similar to the polygon $ABCDEF$ ^e.

P R O P. IV. T H E O R.

A Circle being given, two similar polygons may be found, the one described about the circle and the other inscribed in it, which shall differ from one another by a space less than any given space.

Let ABC be the given circle, and the square of D any given space; a polygon may be inscribed in the circle ABC and a similar polygon described about it, so that the difference between them shall be less than the square of D .

In the circle ABC apply the straight line AE equal to D and let AB be a fourth part of the circumference of the circle. From the circumference AB take away its half, and from the remainder its half, and so on till the circumference AF is found less than the circumference AE ^a. Find the centre G ; draw the diameter AC , as also the straight lines AG and FG ; and having bisected the circumference AF in H join

in KG, and draw HL touching the circle in K, and meeting A and GF produced in H and L; join CF.

Because the isosceles triangles HGL and AGF have the common angle AGF, they are equiangular^b, and the angles GHK, AF are therefore equal to one another. But the angles KH, CFA are also equal, for they are right angles; therefore the triangles HGK, ACF are likewise equiangular^c.

And because the arch AF was found by taking from the arch AB its half, and from that remainder its half, and so AF will be contained a certain number of times exactly the arch AB, and therefore it will also be contained a certain number of times exactly in the whole circumference ABC; and the straight line AF is therefore the side of an equilateral polygon inscribed in the circle ABC. Therefore also, HL is the side of an equilateral polygon of

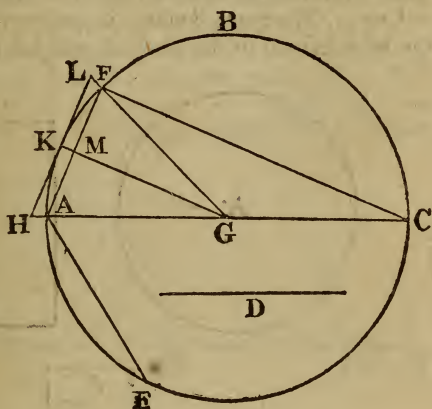
the same number of sides described about the circle ABC^d. Let

the polygon described about the circle be called M, and the polygon inscribed be called N, then, because these polygons are similar^e, they are as

squares of the homologous sides, and AF^f, that is, because the triangles HLG, AFG

are similar, as the side of HG to the

side of AG, that is of GK. But the triangles HGK, ACF have been proved to be similar, and therefore, the square of AC is to the square of CF as the polygon M to the polygon N; and, in conversion, the square of AC is to its excess above the square of CF, that is, to the square of AF^g, as the polygon M to its excess above the polygon N. But the square of AC, that is, the square described about the circle ABC is greater than the square of the equilateral polygon of eight sides described about the circle, because it contains that polygon; and, for the same reason, the square of the equilateral polygon of eight sides is greater than the square of the equilateral polygon of six sides, and so on; therefore, the square of AC is greater than any



d 3. 8.

e cor. 3. 8.

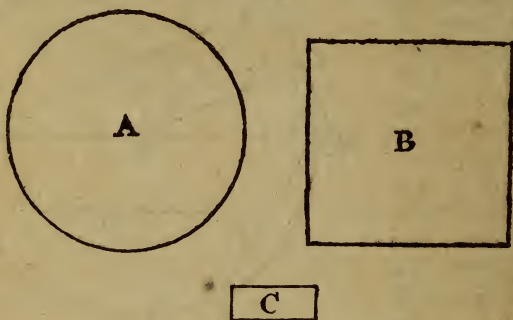
f 3. cor. 29. 6.

g 47. 1.

Book VIII. any polygon described about the circle by the continual section of the arch AB; it is therefore greater than polygon M. Now, it has been demonstrated, that square of AC is to the square of AF as the polygon M to difference of the polygons; therefore, since the square of A is greater than M, the square of AF is greater than the difference of the polygons ^h. The difference of the polygon therefore less than the square of AF; but AF is less than D, therefore, the difference of the polygons is less than the square of D, that is, than the given space. Therefore, &c. Q. E.

COR. 1. Because the polygons M and N differ from the circle by another more than either of them differs from the circle, the difference between each of them and the circle is less than the given space, viz. the square of D. And therefore, however small any given space may be, a polygon may be inscribed in the circle, and another described about it, each of which shall differ from the circle by a space less than the given space.

COR. 2. The space B which is greater than any polygon that can be inscribed in the circle A, and less than any polygon that



can be described about it, is equal to the circle A. If not, let them be unequal; and first, let B exceed A by the space C. Then, because the polygons described about the circle are all greater than B, by hypothesis; and because B is greater than A by the space C, therefore, no polygon can be described about the circle A, but must exceed it by a space greater than C, which is absurd. In the same manner, if B be less than A by the space C, it is shewn that no polygon can be inscribed in the circle A, but is less than A by a space greater than C, which is also absurd. Therefore, A and B are not unequal, that is, they are equal to one another.

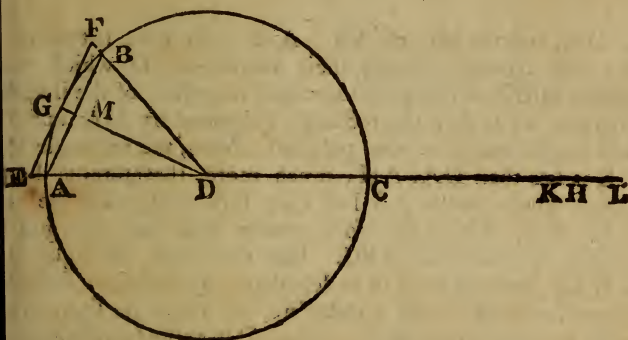
P R O

PROP. V. THEOR.

EVERY circle is equal to the rectangle contained by the semidiameter, and a straight line equal to half the circumference.

Let ABC be a circle, of which the centre is D, and the diameter AC; if in AC produced there be taken AH equal to half the circumference, the circle is equal to the rectangle contained by DA and AH.

Let AB be the side of any equilateral polygon inscribed in the circle ABC; bisect the circumference ABC in G, and through G draw EGF touching the circle, and meeting DA produced in E, and DB produced in F. EF will be the side of an equilateral polygon described about the circle ABC ^a. In AC produced take AK equal to ^a 3. 8. half the perimeter of the polygon whose side is AB; and AL equal to half the perimeter of the polygon whose side is EF. Then AK will be less, and AL greater than the straight line AD ^b. Now, because in the triangle EDF, DG is drawn ^b Ax. 8.



perpendicular to the base, the triangle EDF is equal to the rectangle contained by DG and the half of EF ^c; and as the same ^c 41. 1. is true of all the other equal triangles having their vertices at D, which make up the polygon described about the circle; therefore, the whole polygon is equal to the rectangle contained by S

out the circle ABC, so that its excess above ABC may be less than the square of NP. Let the side of this polygon be EF. And since, as has been proved, the circle is equal to the rectangle DA, AH, and the polygon to the rectangle A, AL, the excess of the polygon above the circle is equal to the rectangle DA, HL; therefore the rectangle DA, HL is less than the square of NP; and therefore, since DA is greater than NP, HL is less than NP, and twice HL less than twice NP, wherefore, much more is twice HL less than NO. But HL is the difference between half the perimeter of the polygon whose side is EF, and half the circumference of the circle; therefore, twice HL is the difference between the whole perimeter of the polygon and the whole circumference of the circle. The difference, therefore, between the perimeter of the polygon and the circumference of the circle is less than the given line NO.

COR. 2. Hence also, a polygon may be inscribed in a circle, such that the excess of the circumference above the perimeter of the polygon may be less than any given line. This is proved like the preceding.

S 2

PROP.

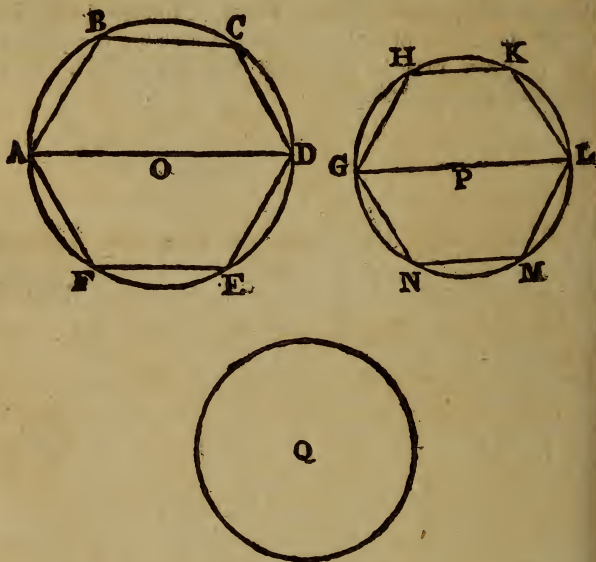


PROP. VI. THEOR.

CIRCLES are to one another in the duplicate ratio, or as the squares of their diameters.

Let ABD and GHL be two circles, of which the diameters are AD and GL; the circle ABD is to the circle GHL as the square of AD to the square of GL.

Let ABCDEF and GHKLMN be two equilateral polygons of the same number of sides inscribed in the circles ABD, GHL; and let Q be such a space that the square AD is to the square of GL as the circle ABD to the space Q.



Because the polygons ABCDEF and GHKLMN are equilateral and of the same number of sides, they are similar ^a, and are as the squares of the diameters of the circles in which they are inscribed. Therefore, as the square of AD to the square of GL, so is the polygon ABCDEF to the polygon GHKLMN.
^a 2. 8.

HKLMN; but as the square of AD to the square of GL, Book VIII.
 is the circle ABD to the space Q; therefore, the polygon
 BCDEF is to the polygon GHKLMN as the circle ABD
 the space Q. But the polygon ABCDEF is less than the
 circle ABD, therefore GHKLMN is less than the space Q^b; ^{b 14. 7.}
 therefore, the space Q is greater than any polygon inscribed
 the circle GHL.

In the same manner it is demonstrated, that Q is less than
 any polygon described about the circle GHL; wherefore, the
 space Q is equal to the circle GHL^c. Now, by hypothesis, ^{c 2. cor. 4.}
 the circle ABD is to the space Q as the square of AD to the ^{8.}
 square of GL; therefore, the circle ABD is to the circle
 GHL as the square of AD to the square of GL. Therefore,
 c. Q. E. D.

COR. I. Hence, the circumferences of circles are to one another
 as their diameters.

Let the straight line X be equal to half the circumference
 of the circle ABD, and the straight line Y to half the cir-
 cumference of the circle GHL: And because the rectangles
 AO, X, and GP, Y are equal to the circles ABD and GHL^d; ^{d 5. 8.}
 therefore the rectangle AO, X is to the rectangle GP, Y as the
 square of AD to the square of GL, or as the square of AO
 the square of

Therefore, **X** _____

alternately the
 rectangle AO,

is to the **Y** _____

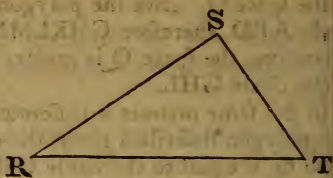
square of AO,

the rectangle GP, Y to the square of GP; but rectangles
 that have equal altitudes are as their bases^e, therefore X is to ^{e 1. 6.}
 AO as Y to GP; and again, alternately, X is to Y as AO
 to GP, and taking the doubles of each, the circumference
 ABD is the circumference GHL as the diameter AD to the
 diameter GL.

Book VIII.

COR. 2. The circle that is described upon the side of right angled triangle opposite to the right angle, is equal to the two circles described on the other two sides.

For the circle described upon SR is to the circle described upon RT as the square of SR to the square of RT; and the circle described upon TS is to the circle described upon RT as the square of ST to the square of RT. Wherefore, the circles described on SR and ST are to the circle described on RT as the squares of SR and ST to the square of RT ^f. But the squares of RS and ST are equal to the square of RT ^g; therefore the circles described on RS and ST are equal to the circle described on RT.



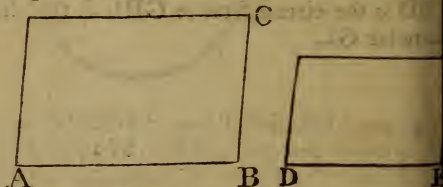
PROP. VII. THEOR.

EQUIANGULAR parallelograms are to one another as the products of the numbers proportional to their sides.

Let AC and DF be two equiangular parallelograms, and let M, N, P and Q be four numbers, such that $AB : BC :: M : N$; $AB : DE :: M : P$, and $AB : EF :: M : Q$, and therefore *ex æquali*, $BC : EF :: N : Q$. The parallelogram AC is to the parallelogram DF as MN to PQ.

Let NP be the product of N into P, and the ratio of MN to PQ

will be compounded of the ratios ^a of MN to NP, and of NP to PQ. But the ratio of MN to

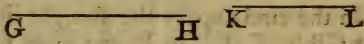


^a 10. def. 5.

^b 15. 5. NP is the same with that of M to P ^b, because MN and NP are equimultiples of M and P; and, for the same reason, the

rat

ratio of NP to PQ is the same with that of N to Q; there-^{Book VIII.}
 fore, the ratio of MN to PQ is compounded of the ratios of ^{by Hyp.}
 M to P, and of N to Q. Now, the ratio of M to P is the
 same with that of the side AB to the side DE^c; and the ratio
 of N to Q the same with that of the side BC to the side EF.
 Therefore the ratio of MN to PQ is compounded of the ra-
 tios of AB to DE, and of BC to EF. And the ratio of the
 parallelogram AC to the parallelogram DF is compounded
 of the same ratios^d; therefore, the parallelogram AC is to ^{d 23. 6.}
 the parallelogram DF as MN. the product of the num-
 bers M and N, to PQ the product of the numbers P and Q.
 therefore, &c. Q. E. D.

COR. 1. Hence, if GH be to KL as the number M to the
 number N; the square
 described on GH will 
 be to the square de-
 scribed on KL as MM the square of the number M to NN
 the square of the number N.

COR. 2. If the square on GH be to the square on KL as a
 number to a number, the line GH will be to the line KL as
 the square root of the first number to the square root of the
 second number. For, if GH were to KL in any other ratio,
 the square described on GH would not be to the square de-
 scribed on KL in the ratio supposed.

Nota. In the following Prop. the character + or — placed after a num-
 ber signifies something is to be added to that number, or something subtracted
 from it. Thus, 1732.0508+ signifies a number greater than 1732.0508, &c.
 Also, it is to be observed, that the figure for distinctness is separated into 2,
 the second being a part of the first magnified or drawn on a greater scale.

S 4

PROP

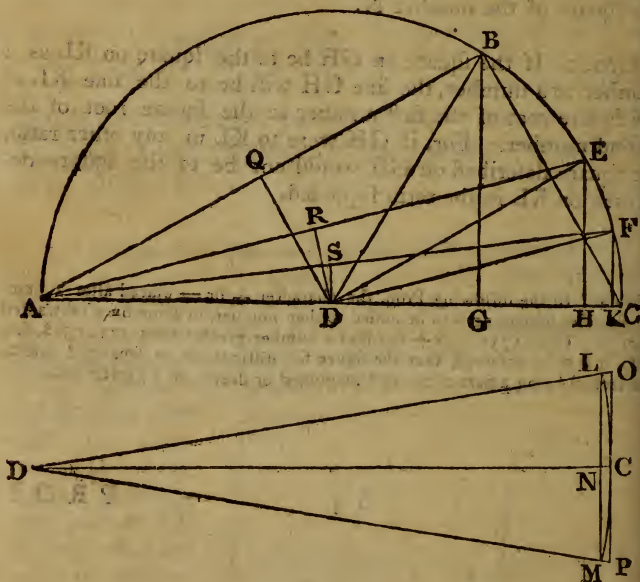
PROP. VIII. THEOR.

See N.

THE circumference of a circle exceeds three times the diameter by a part less than one seventh of the diameter, but greater than ten of the parts whereof the diameter contains seventy-one.

Let ABC be a circle, of which the centre is D, and the diameter AC; the circumference is greater than three times the diameter AC by a line that is less than $\frac{1}{7}$ of AC and greater than $\frac{10}{71}$ of AC.

In the circle apply the straight line CB equal to CD. Bisect CB in E, CE in F, and so on till there be five bisections.



^and let CL, (represented in the 2d figure) be the arch found by the fifth bisection. Take the arch CM equal to CL; join LM.

I, and through C draw OCP touching the circle in C, and meeting DL, DM produced in O and P. Join (fig. 1.) AB, AE, AF, and also DB, DE, DF; draw BG, EH, FK perpendicular to DC, and DQ, DR, DS perpendicular to AB, AE, AF.

Because the straight line CB is equal to CD, the semidiameter of the circle, the arch CB is the sixth part of the circumference ^a; therefore, CE is the 12th part, CF the 24th part; the half of CF the 48th; the half of that half the 96th, and the half of the 96th is that to which CL is equal. CM is equal to CL; and therefore, the arch LM is also the 96th part of the circumference, so that the straight line LM is the side of an equilateral polygon of 96 sides inscribed in the circle, and OP the side of an equilateral polygon of the same number of sides described about the circle ^b. Now, because BD is equal to BC, the angle BDC is equal to the angle BCD, and the angles BGD, BGC are also equal, being right angles; and the side BG is common to both the triangles BGD, BGC; therefore the base DG is equal to the base CG, and DG is the half of DC or AD. Therefore, DG is to DA as 1 to 2; and, by composition, AG to AD as 1 to 2: Now, AC is to AD also as 2 to 1; therefore, the triangle CA, AG is to the square of DA as 6 to 2, or as 3 to 1 ^d. But, because the angle ABC is a right angle, the triangle CA, AG is equal to the square of AB ^e; therefore, the square of AB is to the square of AD as 3 to 1, or 3000000 to 1000000; and therefore, AB is to AD as 32.0508+ to 1000, for the square of 1732.0508 is less than 3000000, and the square of 1000 is equal to 1000000. But Q is the half of AB, because DQ is drawn from the centre perpendicular to AB, therefore also, AQ is to AD as 6.0254+ to 1000.

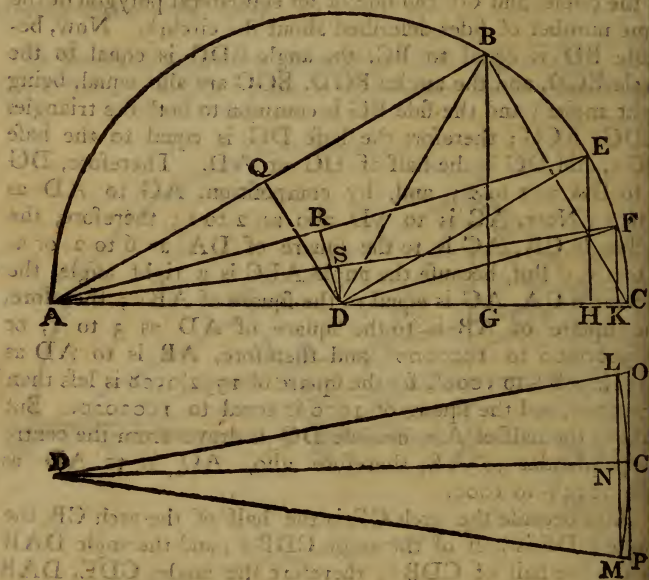
And because the arch CE is the half of the arch CB, the angle CDE is half of the angle CDB ^g; and the angle DAB is also the half of CDB ^h, therefore the angles CDE, DAB are equal. In the triangles DAQ, EDH, therefore, the angle DAQ is equal to the angle EDH, and the angle AQD is equal to the angle DHE, because they are both right angles; and the side AD is also equal to the side DE, therefore the triangle DAQ is equal to the triangle EDH, and the side AQ is equal to the side DH ^c. And it has been shewn, that AQ is to AD as 6.0254+ to 1000, therefore also DH is to AD as

866.0254

Book VIII. 866.0254+ to 1000; and, by composition, AH to AD.
 1866.0254+ to 1000. Therefore also, the rectangle CA

d 7. s. AH is to the square of AD^d as 2000 × 1866.0254 + 1000 × 1000, that is, as 3732050.8 + to 1000000. Wherefore also, the square of AE, which is equal to the rectang

e 8. 6. CA, AH^e , is to the square of AD as $3732050.8 + 1000000$, and therefore, AE is to AD as $1931.8516 + 1000$, for the square of 1931.8516 is less than 3732050 . Therefore, AR, which is the half of AE, or DK, which shewn to be equal to AR in the same manner that DH was proved equal to AQ, is to AD as $965.9258 +$ to 1000.



Hence again, by composition, AK is to AD as 1965.9258 to 1000, and the rectangle CA, AK, or the square of A to the square of AD as 3931851.6+ to 1000000 d, and A to AD as 1982.8897 + to 1000, for the square of 1982.889 is less than 3931851.6.

In the same way it will be shewn, that the straight line drawn from A to the third point of bisection, is to AD as 1995.7178+ to 1000; and therefore, that the square of the

e drawn from A to the fourth point of bisection, is to the Book VIII.
 are of AD as $3995717.8+$ to 1000000 . But DN (fig. 2.)
 equal to half the line from A to the fourth point of bisection,
 n, for L is the fifth point of bisection, and therefore the
 are of DN is the fourth part of the square of that line;
 erefore, the square of DN is to the square of AD, or of
 C as $998929.45+$ to 1000000 . Therefore, by division, the
 are of LN, which is the excess of the square of DC, or
 DL, above the square of DN, is to the square of CD as
 e excess of 1000000 above $998929.45+$ is to 1000000 , that
 as $1070.55-$ to 1000000 . Therefore also, LN has to
 D the ratio of $32.71927-$ to 1000 ; for the square of
 32.71927 is greater than 1070.55 .

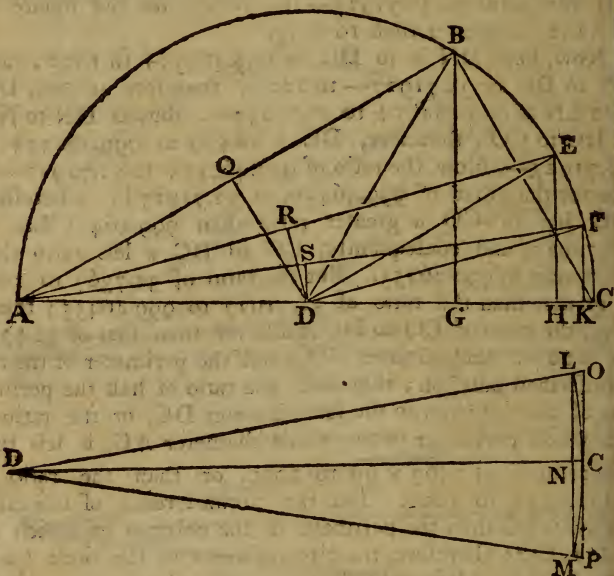
Now, since DN is to DC as $999.46455+$ to 1000 ; and
 N to DC as $32.71927-$ to 1000 ; therefore, *ex equo*, DN
 to LN as $999.46455+$ to $32.71927-$. But, as DN to NL,
 DC to CO; therefore, DC is to CO as $999.46455+$ to
 $32.71927-$. Now, the ratio of $999.46455+$ to $32.71927-$ is
 eater than that of 999.46455 to 32.71927^1 ; wherefore. l 8. 5.
 C has to CO a greater ratio than 999.46455 has to
 32.71927 , and consequently, CO to DC a less ratio than
 32.71927 to 999.46455 . But the ratio of 32.7367 to 1000
 greater than the ratio of 32.71927 to 999.46455 ; there-
 re, the ratio of CO to DC is also less than that of 32.7367
 1000 m. And 96 times CO is half the perimeter of the cir- m 13. 5.
 umscribed polygon; therefore, the ratio of half the perime-
 r of that polygon to the semidiameter DC, or the ratio of
 e whole perimeter to the whole diameter AC, is less than
 e ratio of 32.7367×96 to 1000 , or than the ratio of
 3142.7232 to 1000 . But the circumference of the circle
 BC is less than the perimeter of the polygon of which the
 de is CO; therefore, the circumference of the circle has to
 e diameter a less ratio than 3142.7232 has to 1000 . Now,
 e ratio of 22 to 7 is greater than that of 3142.7232 to
 1000 , therefore, the circumference of the circle ABC has to
 e diameter a less ratio than 22 has to 7; and therefore, the
 art by which the circumference exceeds three times the dia-
 meter is less than a seventh of the diameter.

It remains to demonstrate, that the part by which the cir-
 cumference exceeds three times the diameter is greater than
 o
 of the diameter.

Beauf

Book VIII.

Because, as was already shewn, the square of AB is to the square of AD as 3000000 to 1000000, AB is to AD as 1732.0509— to 1000; for the square of 1732.0509 is greater than 3000000. Therefore, the half of AB, that is, AC or DH is to DA as 866.02545— to 1000, and, by composition, AH to AD as 1866.02545— to 1000; and therefore the rectangle CA, AH, that is, the square of AE is to the square of AD as 3732050.9— to 1000000; and therefore also, AE is to AD as 1931.8517— to 1000; for the square of 1931.8517 is greater than 3732050.9. Therefore also, AF



or DK, which is equal to AR, is to AD as 965.92585— to 1000, and, by composition, AK is to AD as 1965.92585— to 1000.

Hence, the rectangle CA, AK, that is, the square of AF is to the square of AD as 3931851.7—to 1000000; and therefore, AF to AD as 1982.8898—to 1000, for the square of 1982.8898 is greater than 3931851.7.

In like manner it is shewn, that the straight line drawn from A to the third point of bisection is to AD as 1995.7179—

or 1000; and that, therefore, the square of the line drawn from A to the fourth point of bisection, is to the square of AD as 3995717.9— to 1000000. But DN is the half of the straight line drawn from A to the fourth point of bisection, because L is the fifth point of bisection, and therefore the square of DN is the fourth part of the square of the line drawn to the point of the fourth bisection; therefore the square of DN is to the square of AD, or of DC, as 98929.475— to 1000000. Wherefore also, the excess of the square of DC or of DL above the square of DN, that is, the square of LN is to the square of DC as the excess of 1000000 above 98929.475—, or as 1070.525+ to 1000000; and therefore, LN is to DC as 32.71887+ to 1000, for the square of 32.71887 is less than 1070.525. Therefore, LN is to DC a greater ratio¹ than 32.71887 to 1000, and 96 times LN, or half the perimeter of the inscribed polygon of 6 sides, has to the semidiameter DC a greater ratio than that of 32.71887×96 to 1000; therefore also, the whole perimeter of the polygon has to the whole diameter AC a greater ratio than 32.71887×96, or than 3141.01252 has to 1000. Now, the circumference of the circle is greater than the perimeter of the inscribed polygon, therefore the circumference of the circle has to the diameter a greater ratio than 3141.01252 has to 1000. But the ratio of 3141.01252 to 1000 is greater than that of 3 and $\frac{10}{71}$ to 1. Wherefore, the excess of the circumference above three times the diameter is greater than 10 of those parts whereof the diameter contains 71; and it has already been shewn to be less than one seventh of the diameter. Therefore, &c. Q. E. D.

COR. 1. Hence, the diameter of a circle being given, the circumference may be found nearly, by making as 7 to 22, so the diameter to a fourth proportional, which will be greater than the circumference. And, if as 1 to 3 and $\frac{10}{71}$ so the diameter to a fourth proportional, this will also be nearly equal to the circumference, but will be less than it.

Book VIII.

COR. 2. Because the difference between $\frac{1}{7}$ and $\frac{10}{71}$ is $\frac{1}{497}$

therefore the lines found by these proportions differ by $\frac{1}{497}$ the diameter. Therefore, the difference of either of the from the circumference must be less than the 497th part the diameter.

COR. 3. The ratio of a circle to the square of its diameter is nearly that of 11 to 14. For the ratio of the circumference to the diameter is that of 22 to 7 nearly, and therefore, the ratio of half the circumference to the diameter, is nearly that of 11 to 7; and the ratio of half the diameter to the whole is that of 1 to 2; therefore, the ratio compounded of these, that is, of the rectangle under half the circumference and half the diameter, to the square of the diameter is that of 11 to 14, nearly. But the rectangle under half the circumference and half the diameter is equal to the circle; therefore, the circle is to the square of the diameter as 11 to 14 nearly.

n 5. 8.

COR. 4. It is evident, that the method used in this proposition for finding the limits of the ratio of the circumference to the diameter may be carried to a greater degree of exactness, by finding the perimeter of an inscribed, and a circumscribed polygon of a greater number of sides than 96. Thus by inscribing in the circle a polygon of 40960 sides, and describing about it another of the same number of sides, it may be demonstrated, that the ratio of the circumference to the diameter is greater than that of 314159265 to 100000000, but less than that of 314159266 to 100000000.

PRO P

PROP. IX. THEOR.

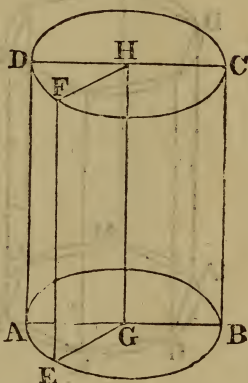
From any point in the circumference of the base of a cylinder a straight line be drawn perpendicular to the plane of the base, it will be wholly in the cylindric superficies.

Let ABCD be a cylinder, of which the base is the circle AEB, DFC the circle opposite to the base, and GH the axis; from E, any point in the circumference AEB, let EF be drawn perpendicular to the plane of the circle AEB; the straight line EF is in the superficies of the cylinder.

Let F be the point in which EF meets the plane DFC opposite to the base; join EG and FH; and let AGHD be the rectangle^a by the revolution of which the cylinder ABCD is described.

Now, because GH is at right angles to GA, the straight line which by its revolution describes the circle AEB, it is at right angles to all the straight lines in the plane of that circle which meet it at G, and it is therefore at right angles to the plane of the circle AEB. But EF is at right angles to the same plane; therefore, EF

and GH are parallel^b, and in the same plane. And since the line through GH and EF cuts the parallel planes AEB, DFC in the straight lines EG and FH, EG is parallel to FH^c. The figure EGHF is therefore a parallelogram, and it has the angle EGH a right angle, therefore it is a rectangle, and is equal to the rectangle AH, because EG is equal to AG. Therefore, when in the revolution of the rectangle AH, the straight line AG coincides with EG, the two rectangles AH and EH will coincide, and the straight line AD will coincide



^a 1. def. 7.

^b 6. 7.

^c 14. 7.

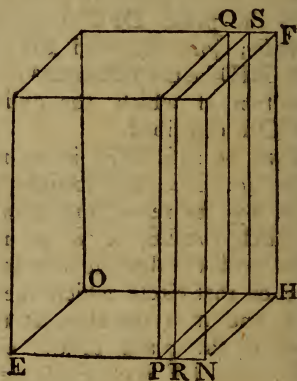
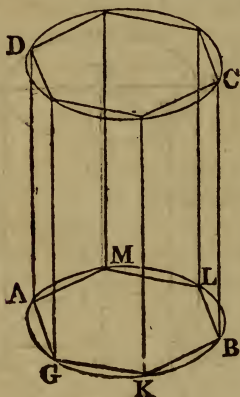
Book VIII.

incide with the straight line EF. But AD is always in the superficies of the cylinder, for it describes that superficies; therefore, EF is also in the superficies of the cylinder. Therefore, &c. Q. E. D.

PROP. X. THEOR.

A CYLINDER and a parallelepiped having equal bases and altitudes are equal to one another.

Let ABCD be a cylinder, and EF a parallelepiped having equal bases, viz. the circle AGB and the parallelogram EF, and having also equal altitudes; the cylinder ABCD is equal to the parallelepiped EF.



If not, let them be unequal; and first, let the cylinder be less than the parallelepiped EF; and from the parallelepiped EF let there be cut off a part EQ by a plane PQ parallel to NF, equal to the cylinder ABCD. In the circle AGB inscribe the polygon AGKBLM that shall differ from the circle by a space less than the parallelogram PH, and cut off from the parallelogram EH, a part OR equal to the polygon AGKBLM. The point R will fall between P and N. Of the polygon AGKBLM let an upright prism AGBCD be constituted of the same altitude with the cylinder, which will

therefore

before be less than the cylinder, because it is within it ^b; Book VIII.
 if through the point R a plane RS parallel to NF be b 9. 8.
 de to pass, it will cut off the parallelepiped ES equal to
 prism AGBC, because its base is equal to that of the
 m, and its altitude is the same. But the prism AGBC is
 than the cylinder ABCD, and the cylinder ABCD is e-
 l to the parallelepiped EQ, by hypothesis; therefore, ES
 less than EQ, and it is also greater, which is impossible.
 e cylinder ABCD, therefore, is not less than the paral-
 piped EF; and in the same manner it may be shewn
 to be greater than EF. Therefore they are equal.
 E. D.

T

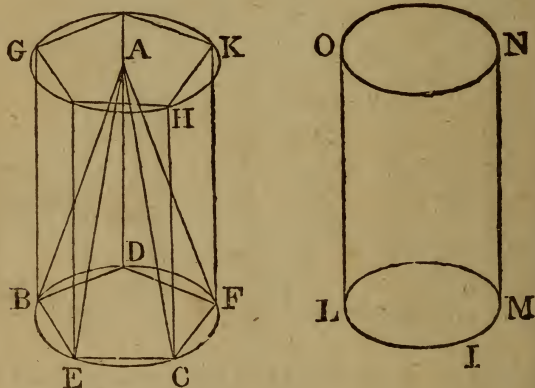
P R P.

PROP. XI. THEOR.

IF a cone and a cylinder have the same base and the same altitude, the cone is the third part of the cylinder.

Let the cone $ABCD$, and the cylinder $BFKG$ have same base, viz. the circle BCD , and the same altitude, viz. perpendicular from the point A upon the plane BCD , cone $ABCD$ is the third part of the cylinder $BFKG$.

If not, let the cone $ABCD$ be the third part of another cylinder $LMNO$, having the same altitude with the cylinder $BFKG$, but let the bases BCD and LIM be unequal; first, let BCD be greater than LIM .



Then because the circle BCD is greater than the circle LIM , a polygon may be inscribed in BCD , that shall differ from it less than LIM does, and which, therefore, will be greater than LIM . Let this be the polygon $BECFD$ and upon $BECFD$ let there be constituted the pyramid $ABECFD$, and the prism $BCFKHG$.

Because the polygon $BECFD$ is greater than the circle LIM , the prism $BCFKHG$, is greater than the cylinder $LMNO$.

INO, for they have the same altitude, but the prism has ^{Book VIII.} greater base. But the pyramid ABECFD is the third ^b 35. 7. part of the prism ^b BCFKHG, therefore, it is greater than the third part of the cylinder LMNO. Now, the cone BECFD is, by hypothesis, the third part of the cylinder INO, therefore, the pyramid ABECFD is greater than cone ABCD, and it is also less, because it is inscribed in cone LMNO, which is impossible. Therefore, the cone ABCD is not less than the third part of the cylinder BFKG: And, in the same manner, by circumscribing a polygon about the circle BCD, it may be shewn, that the cone ABCD is not greater than the third part of the cylinder BFKG; therefore, it is equal to the third part of that cylinder. Q. E. D.

P R O P. XII. T H E O R.

If a hemisphere and a cone have equal bases and altitudes, a series of cylinders may be inscribed in the hemisphere, and another series may be described about the cone, having all the same altitudes with one another, and such that their sum shall differ from the sum of the hemisphere, and cone by a solid less than any given solid.

Let ADB be a semicircle, of which the centre is C, and CD be at right angles to AB; let DB and DA be arcs described on DC, draw DE and let the figure so constructed revolve about DC: then, the sector BCD, which is the half of the semicircle ADB, will describe a hemisphere having C for its centre ^a 14. def. and the triangle CDE will describe a cone, having its vertex at C, and having for its base the circle ^b described by DE, equal to that described by ^b 18. def. which is the base of the hemisphere. Let W be any given solid. A series of cylinders may be inscribed in the hemisphere ADB, and another described about the cone ECL, that their sum shall differ from the sum of the hemisphere and the cone, by a solid less than the solid W.

Upon the base of the hemisphere let a cylinder be constructed equal to W, and let its altitude be CX. Divide CD into

inders and the sum of the hemisphere and the cone, is e- Book VIII.
 al to the difference of two solids, which are each of them
 s than W; but this difference must also be less than W,
 erefore the difference between the two series of cylinders
 d the sum of the hemisphere and cone is less than the
 en solid W. Q. E. D.

P R O P. XIII.

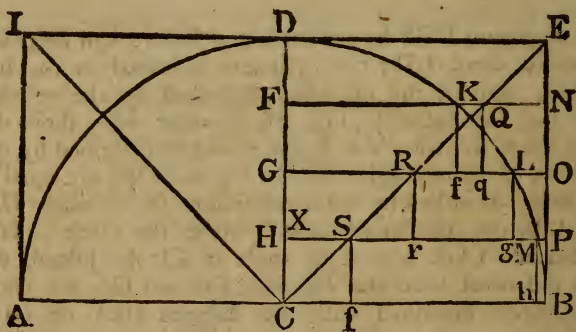
THE same things being supposed as in the last
 proposition, the sum of all the cylinders in-
 scribed in the hemisphere, and described about the
 ne, is equal to a cylinder, having the same base
 d altitude with the hemisphere.

Let the figure DCB be constructed as before, and supposed
 revolve about CD; the cylinders inscribed in the he-
 msphere, that is, the cylinders described by the revolu-
 n of the rectangles Hh, Gg, Ff, together with those de-
 igned about the cone, that is, the cylinders described by the
 volution of the rectangles Hs, Gr, Fq, and DN, are equal to
 e cylinder described by the revolution of the rectangle DB.
 Let L be the point in which GO meets the circle ADB,
 en, because CGL is a right angle if CL be joined, the
 cles described with the distances CG and GL are equal
 the circle described with the distance CL^a or GO; a 2. cor. 6.
 w CG is equal to GR, because CD is equal to DE, 8.
 d therefore also, the circles described with the distances
 R and GL are together equal to the circle described
 th the distance GO, that is, the circles described by
 e revolution of GR and GL about the point G, are to-
 ther equal to the circle described by the revolution of GO
 out the same point G; therefore also, the cylinders that
 nd upon the two first of these circles having the common
 itude GH, are equal to the cylinder which stands on the re-
 maining circle, and which has the same altitude GH. The cy-
 nders described by the revolution of the rectangles Gg and Gr
 e therefore equal to the cylinder described by the rectangle
 P. And as the same may be shewn of all the rest, there-
 e the cylinders described by the rectangles Hh, Gg, Ff,
 d by the rectangles Hs, Gr, Fq, DN, are together equal to
 e cylinder described by DB, that is, to the cylinder having
 e same base and altitude with the hemisphere. Q. E. D.

P R O P. XIV.

EVERY sphere is two thirds of the circumscribing cylinder.

Let the figure be constructed as in the two last propositions, and if the hemisphere described by BDC be not equal to two thirds of the cylinder described by BD, let it be greater by the solid W. Then, as the cone described by CDE is one third of the cylinder^a described by BD, the cone and the hemisphere together will exceed the cylinder by W. But



- ^b 13. 8. that cylinder is equal to the sum of all the cylinders described by the rectangle Hh, Gg, Ff, Hs, Gr, Fq, DN^b; therefore the hemisphere and the cone added together exceed the sum of all these cylinders by the given solid W;
- ^c 12. 8. which is absurd, for it has been shewn^c, that the hemisphere and the cone together differ from the sum of the cylinders by a solid less than W. The hemisphere is therefore equal to two thirds of the cylinder described by the rectangle BD; and therefore the whole sphere is equal to two thirds of the cylinder described by twice the rectangle BD, that is, to two thirds of the circumscribing cylinder. Q. E. D.

FINIS.

E L E M E N T S

OF

P L A N E

AND

S P H E R I C A L

TRIGONOMETRY.

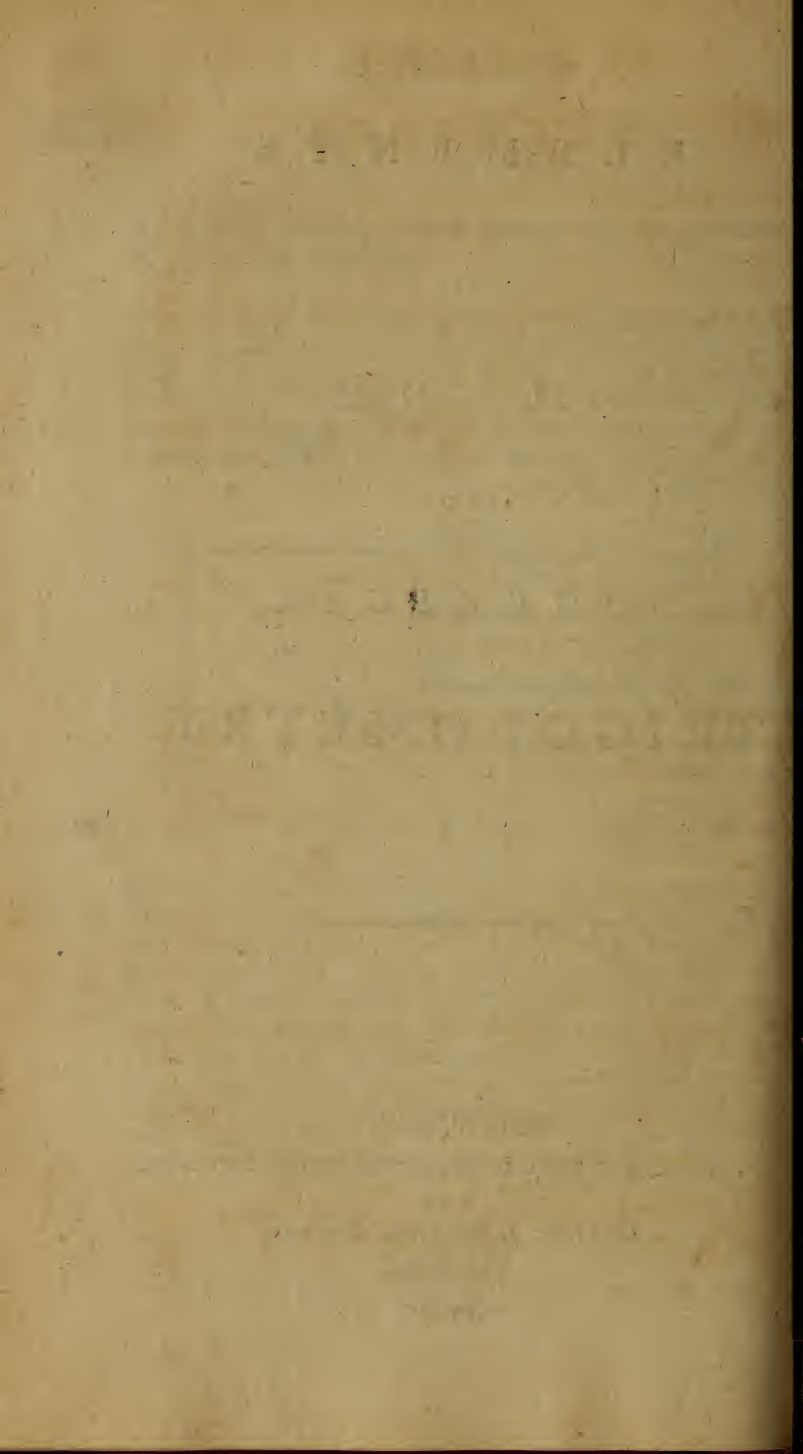
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ELEMENTS

OF

PLANE TRIGONOMETRY.



LEMMA I.

AN angle at the centre of a circle is to four right angles as the arch on which it stands is to the whole circumference.

Let ABC be an angle at the centre of the circle ACF , standing on the circumference AC : the angle ABC is to four right angles as the arch AC to the whole circumference ACF .

Produce AB till it meet the circle in E , and draw DBF perpendicular to AE .

Then, because ABC , BD are two angles at the centre of the circle ACF , the angle ABC to the angle ABD as the arch AC to the arch AD , (36. 6.); and therefore also, the angle ABC is to four times the angle ABD as the arch AC to four times the arch AD , (4. 5.).

But ABD is a right angle, and therefore, four times the angle ABD is equal to the whole circumference ACF ; therefore,

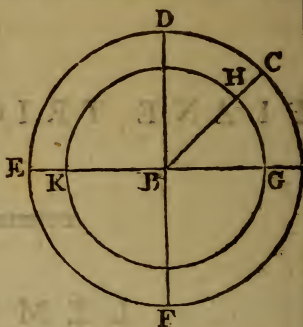


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fore,

fore, the angle ABC is to four right angles as the arch AC to the whole circumference ACF .

COR. Equal angles at the centre of different circles stand on arches which have the same ratio to their circumferences. For, if the angle ABC , at the centre of the circles ACE , GHK , stand on the arches AC , GH , AC is to the whole circumference of the circle ACE , as the angle ABC to four right angles; and the arch HG is to the whole circumference of the circle GHK in the same ratio. Therefore, &c.



DEFINITIONS.

I.

If two straight lines intersect one another in the centre of a circle, the arch of the circumference intercepted between them is called the measure of the angle which they contain. Thus, the arch AC is the measure of the angle ABC .

II.

If the circumference of a circle be divided into 360 equal parts, each of these parts is called a degree; and, if a degree be divided into 60 equal parts, each of these is called a minute; and, if a minute be divided into 60 equal parts, each of them is called a second, and so on. And as many degrees, minutes, seconds, &c. as are in any arch, so many degrees, minutes, seconds, &c. is said to be in the angle measured by that arch.

COR. 1. Any arch is to the whole circumference of which it is a part, as the number of degrees, and parts of a degree contained in it, is to the number 360. And any angle is

for

VI.

A straight line AE touching the circle at A , one extremity of the arch AC , and meeting the diameter BC passing through the other extremity C in E , is called the *Tangent* of the arch AC , or of the angle ABC .

COR. The tangent of half a right angle is equal to the radius.

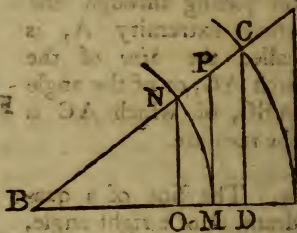
VII.

The straight line BE between the centre and the extremity of the tangent AE , is called the *Secant* of the arch AC or of the angle ABC .

COR. to Def. 4. 6. 7. the sine, tangent, and secant of an angle ABC , are likewise the sine, tangent, and secant of supplement CBF .

It is manifest from Def. 4. that CD is the sine of the angle CBF . Let CB be produced till it meet the circle again I ; and it is manifest, that AE is the tangent, and BE the secant, of the angle ABI or CBF , from def. 8. 7.

COR. to Def. 4. 5. 6. 7. The sine, versed sine, tangent, and secant of an arch, which is the measure of any given angle ABC is to the sine, versed sine, tangent and secant, of any other arch which is the measure of the same angle, as the radius of the first arch is to the radius of the second.



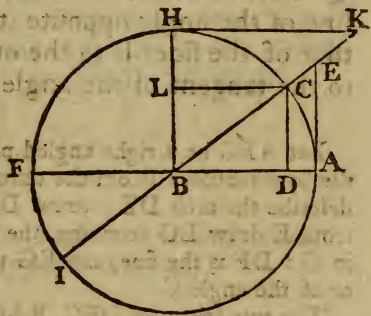
Let AC , MN be measures of the angle ABC , according Def. 1. CD the sine, DA the versed sine, AE the tangent and BE the secant of the arch AC , according to def. 4. 6. 7. and NO the sine, OM the versed sine, MP the tangent, and BP the secant of the arch MN , according to the same definitions.

same definitions. Since CD , NO , AE , MP are parallel, CD is to NO as the radius CB to the radius NB , and AE to MP as AB to BM , and BC or BA to BD as BN or BM to BO ; and, by conversion, DA to MO as AB to MB . Hence the corollary is manifest. And therefore, if tables be constructed, exhibiting in numbers the sines, tangents, secants, and versed sines of certain angles to a given radius, they will exhibit the ratios of the sines, tangents, &c. of the same angles to any radius whatsoever.

In such tables, which are called trigonometrical tables, the radius is either supposed 1, or some number in the series 10, 100, 1000, &c. The use and construction of these tables will appear from what follows.

VIII.

The difference between any angle and a right angle, or between any arch and a quadrant, is called the complement of that angle, or of that arch. Thus, if BH be perpendicular to AB; the angle CBH is the complement of the angle ABC, and the arch HC the complement of AC; also the complement of the obtuse angle FBC is the angle HBC, its excess above a right angle; and the complement of the arch FC is HC.



IX.

The sine, tangent, or secant of the complement of any angle is called the cosine, cotangent, or cosecant of that angle. Thus, let CL or DB, which is equal to CL, be the sine of the angle CBH; HK the tangent, and BK the secant of the same angle; CL or BD is the cosine, HK the cotangent, and BK the cosecant of the angle ABC.

COR. 1. The radius is a mean proportional between the tangent and co-tangent of any angle ABC .

For, since HK , BA are parallel, the angles HKB , ABC are equal, and KHB , BAE are right angles; therefore the triangles BAE , KHB are similar, and therefore AE to AB , as BH or BA to HK .

COR. 2. The radius is a mean proportional between the cosine and secant of any angle ABC .

Since CD , AE are parallel, BD is to BC or BA , as BA to BE .

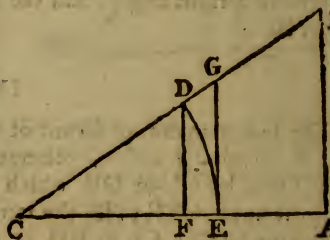
PROP. I.

IN a right angled plain triangle, as the hypotenuse is to either of the sides, so the radius is to the sine of the angle opposite to that side; and as either of the sides is to the other side, so is the radius to the tangent of the angle opposite to that side.

Let ABC be a right angled plane triangle, of which BC the hypotenuse. From the centre C , with any radius CD describe the arch DE ; draw DF at right angles to CE , and from E draw EG touching the circle in E , and meeting CA in G : DF is the sine, and EG the tangent of the arch DE or of the angle C .

The two triangles DFC , BAC are equiangular, because the angles DFC , BAC are right angles, and the angle at C is common. Therefore, $CB : BA :: CD : DF$; but CD is the radius, and DF the sine of the angle C (def. 4.), therefore $CB : BA :: R : \sin. C$.

Also, because EG touches the circle in E , CEG is a right angle, and therefore equal to the angle BAC ; and since the angle at C is common to the triangle

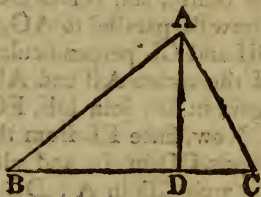


$\triangle CBA$

EBA, CGE, these triangles are equiangular, wherefore $CE : EG :: CA : AB$; but CE is the radius, and EG the tangent of the angle C; therefore, $CA : AB :: R : \tan. C$.

COR. 1. As the radius to the secant of the angle C, so the side adjacent to that angle to the hypotenuse. For CG is the secant of the angle C (def. 7.), and the triangles CGE, EBA being equiangular, $CA : CB :: CE : CG$, that is, $CA : CB :: R : \sec. C$.

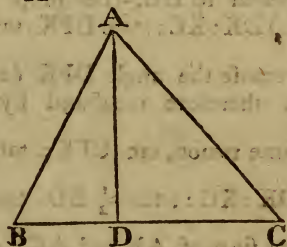
COR. 2. In every triangle, if a perpendicular be drawn from any of the angles on the opposite side, the segments of that side are to one another, as the tangents of the parts into which the opposite angle is divided by the perpendicular. For, if in the triangle ABC, AD be drawn perpendicular to the base BC, each of the triangles CAD, ABD being right angled, $AD : DC :: R : \tan. CAD$, and $AD : DB :: R : \tan. DAB$; therefore, *ex æquo*, $DC : DB :: \tan. CAD : \tan. BAD$.



PROP. II.

THE sides of a plain triangle are to one another as the sines of the opposite angles.

From A any angle in the triangle ABC, let AD be drawn perpendicular to BC. And because the triangle ABD is right angled at D, $AB : AD :: R : \sin. B$; and, for the same reason, $AC : AD :: R : \sin. C$, and inferrely, $AD : AC :: \sin. C : R$; therefore, *ex æquo perturbatè*, $AB : AC :: \sin. C : \sin. B$. In the same manner, it may be demonstrated, that $AB : BC :: \sin. C : \sin. A$. Therefore, &c. Q. E. D.

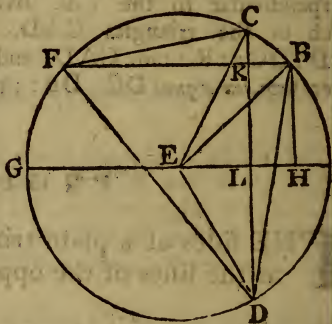


PROP. III.

THE sum of the sines of any two arches of a circle, is to the difference of their sines, as the tangent of half the sum of the arches to the tangent of half their difference.

Let AB, AC be two arches of a circle $ABCD$; let E be the centre, and AEG the diameter which passes through E . Draw BF parallel to AG , meeting the circle again in F . Draw BH and CL perpendicular to AE , and they will be the sines of the arches AB and AC ; produce CL till it meet the circle again in D ; join DF, FC, DE, EB, EC .

Now, since EL from the centre is perpendicular to CD , it bisects CD in L , and also the arch CD in A ; DL is therefore equal to the sine of the arch AC , and BH or LK being the sine of AB , DK is the sum of the sines of the arches AC and AB , and CK the difference of their sines: DB also is the sum of the arches AC and AB , because AD is equal to AC , and BC is their difference. Now, in the triangle DFC , because FK is perpendicular to DC , (cor. prop. 1.) $DK : KC :: \tan. DFK : \tan. CFK$; but $\tan. DFK = \tan. \frac{1}{2} DB$



because the angle DFK (20. 3.) is the half of DEB , and is therefore measured by half the arch DB . For the

same reason, $\tan. CFK = \tan. \frac{1}{2} BC$; and consequently,

$DK : KC :: \tan. \frac{1}{2} BD : \tan. \frac{1}{2} BC$. But DK is the sum of the sines of AB and AC ; KC the difference of their sines BD is the sum of AB and AC , and BC their difference. Therefore, &c. Q. E. D.

COR. 1. Because EL is the cosine of AC, and EH of AB, FK is the sum of these cosines, and KB their difference. Now, $FK : KB :: \tan. FDK : \tan. BDK$; and $\tan. FDK = \text{co-tan. DFK}$, because DFK is the complement of FDK; therefore, $FK : KB :: \text{co-tan. DFK} : \tan. BDK$;

that is, $FK : KB :: \text{co-tan. } \frac{1}{2} DB : \tan. \frac{1}{2} BC$. The sum of the cosines of two arches is therefore to the difference of the same cosines, as the co-tangent of half the sum of the arches to the tangent of half their difference.

COR. 2. If the two arches AB and AC be together equal to 90° , the tangent of half their sum, that is, of 45° , is equal to the radius. And the arch BC being the excess of DC above DB, or above 90° , the half of the arch BC will be equal to the excess of the half of DC above the half of DB, that is, to the excess of AC above 45° ; therefore, when the sum of two arches is 90° , the sum of the sines of those arches is to their difference as the radius to the tangent of the difference between either of them and 45° .



P R O P.

PROP. IV.

THE sum of the sides of any plane triangle is to their difference, as the tangent of half the sum of the angles at the base of the triangle, to the tangent of half their difference.

Let ABC be any plain triangle;

$$CA + AB : CA - AB :: \tan. \frac{B+C}{2} : \tan. \frac{B-C}{2}.$$

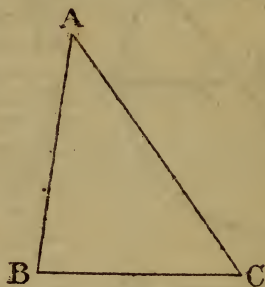
For (Prop. 2.) $CA : AB :: \sin. B : \sin. C$;
and therefore, by composition and division,

$$CA + AB : CA - AB :: \sin. B + \sin. C : \sin. B - \sin. C.$$

But, by the last, $\sin. B + \sin. C : \sin. B - \sin. C ::$

$$\tan. \frac{B+C}{2} : \tan. \frac{B-C}{2}; \text{ therefore also,}$$

$$CA + AB : CA - AB :: \tan. \frac{B+C}{2} : \tan. \frac{B-C}{2}. \quad Q. E. D.$$



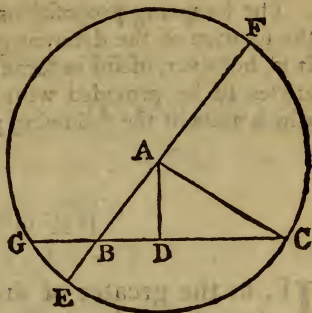
PROP. V.

IN a plane triangle, if the perpendicular drawn to the base from the opposite angle fall within the triangle, the base is to the sum of the sides as the difference of the sides to the difference of the segments of the base; but, if the perpendicular fall without the triangle, the base is to the sum of the sides as the difference of the sides to the sum of the segments of the base.

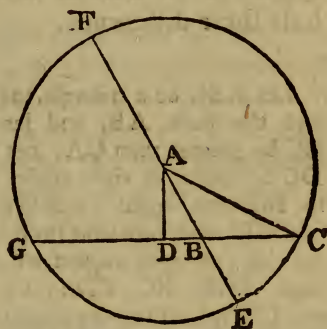
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Let ABC be a plane triangle, and AD the perpendicular to the base from the opposite angle A .

1. Let AD be within the triangle. From A as a centre, with the distance AC , the greater of the two sides, let a circle be described meeting AB produced in E and F , and CB in G . It is manifest, that FB is the sum, and BE the difference of the sides; and, since AD is drawn from the centre perpendicular to CG , CD is equal to DG , and therefore, BG is the difference of DC and DB , that is, of the segments of the base. Now, the rectangle CBG is equal to the rectangle FBE (35. 3.); and therefore, $CB : BF :: BE : BG$ (16. 6.), that is, the base is to the sum of the sides, as the difference of the sides to the difference of the segments of the base.



2. Let AD be without the triangle. It will be demonstrated as in the last case, that $CB : BF :: BE : BG$; now, BG is equal to GD and DB , that is, to CD and DB , or to the sum of the segments of the base; therefore, when the perpendicular falls without the triangle, the base is to the sum of the sides, as the difference of the sides to the sum of the segments of the base. Q. E. D.



COR. The perpendicular AD falls within the triangle, when the square of the greater side AC is less than the sum of the squares on the other side, and on the base (13. 2.); and, when the contrary happens, the perpendicular falls without the triangle. Therefore, according as the square on AC is less or greater than the squares on AB and BC , the triangle falls under the first or second case of this proposition.

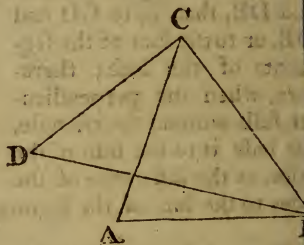
SCHOLIUM.

The foregoing propositions contain all that is necessary for the solution of the different questions in plane trigonometry. It is, however, useful in some of the cases of oblique angled triangles, to be provided with other methods of calculation; which account the following propositions are added.

PROP. VI.

IF, as the greater of any two sides of a triangle to the less, so the radius to the tangent of a certain angle; then will the radius be to the tangent of the difference between that angle and half right angle, as the tangent of half the sum of the angles at the base of the triangle to the tangent half their difference.

Let ABC be a triangle, the sides of which are BC and CA, and the base AB, and let BC be greater than CA. Let DC be drawn at right angles to BC, and equal to AC; join BD, and because (Prop. 1.) in the right angled triangle BCD, $BC : CD :: R : \tan. CBD$, CBD is the angle, of which the tangent is to the radius as CD to BC, that is, as CA to BC, or as the least of the two sides of the triangle to the greatest.



But $BC + CD : BC - CD :: \tan. \frac{1}{2} (CDB + CBD) :$

$\tan. \frac{1}{2} (CDB - CBD),$ (Prop. 5.);

and also, $BC+CA : BC-CA :: \tan. \frac{1}{2} (CAB+CBA) : \tan. \frac{1}{2} (CAB-CBA)$. Therefore, since $CD=CA$,
 $\tan. \frac{1}{2} (CDB+CBD) : \tan. \frac{1}{2} (CDB-CBD) :: R : \tan. \frac{1}{2} (CAB+CBA) : \tan. \frac{1}{2} (CAB-CBA)$. But, because
the angles $CDB+CBD = 90^\circ$, $\tan. \frac{1}{2} (CDB+CBD) = R$.
 $\tan. \frac{1}{2} (CDB-CBD) :: R : \tan. (45^\circ - CBD)$, (2. cor.
Prop. 3.); therefore, $R : \tan. (45^\circ - CBD) :: \tan. \frac{1}{2} (CAB+CBA) : \tan. \frac{1}{2} (CAB-CBA)$; and CBD
was already shewn to be such an angle that
 $BC : CA :: R : \tan. CBD$. Therefore, &c. Q. E. D.

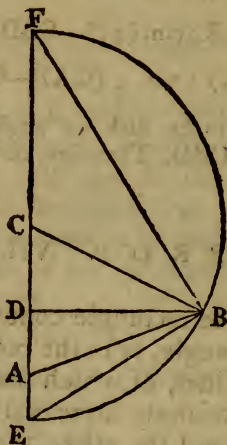
P R O P. VII.

FOUR times the rectangle contained by any two
sides of a triangle, is to the rectangle contained
by two straight lines, of which one is the base or
third side of the triangle increased by the difference
of the two sides, and the other the base diminished
by the difference of the same sides, as the square of
the radius to the square of the sine of half the angle
included between the two sides of the triangle.

Let ABC be a triangle, of which AB is the base, and CB
the greater of the two sides. From B draw BD at right
angles to AC .

Case 1. When ACB is an acute angle, so that BD falls
within the triangle. From the centre C , with the distance
 CB , describe the circle EBF , meeting AC produced in E and
 F . Join BE , BF ; and it is evident, that EBF is a right
angle, and also, that the angle EFB is half the angle ECB .

Then, (13. 2.) $AC^2 + CB^2$, that is,
 $AC^2 + CE^2 = AB^2 + 2AC.CD$. But, (7. 2.)
 $AC^2 + CE^2 = 2AC.CE + AE^2$, therefore,
 $2AC.CE + AE^2 = AB^2 + 2AC.CD$; and if the rectangle
 $2AC.CD$ be taken from both, $2AC.CE - 2AC.CD + AE^2$
 $= AB^2$, and therefore, because
 $2AC.CE - 2AC.CD = 2AC.ED$, $2AC.ED + AE^2 = AB^2$
or $2AC.ED = AB^2 - AE^2$. But (5. 2.)
 $AB^2 - AE^2 = (AB + AE)(AB - AE)$, therefore,
 $2AC.ED = (AB + AE)(AB - AE)$.



Again, $AC : CE :: 2AC.DE : 2CE.DE$, because rectangles that have equal altitudes are as their bases; and, for the same reason, $4AC.CE : 4CE^2 :: 2AC.DE : 2CE.DE$, or alternately, $4AC.CE : 2AC.DE :: 4CE^2 : 2CE.DE$. Now, $4CE^2 = EF^2$, and $2CE.DE = FE.DE = EB^2$ (cor. 8. 6.), therefore, $4AC.CE : 2AC.DE :: EF^2 : EB^2$. But

$EF : EB :: R : \sin. EFB :: R : \sin. \frac{1}{2} C$, whence,

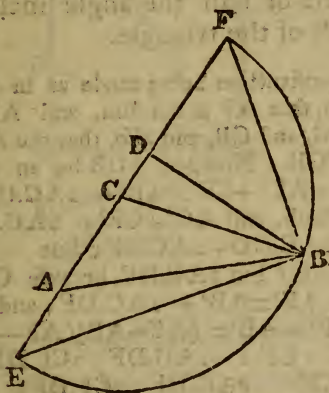
$EF^2 : EB^2 :: R^2 : (\sin. \frac{1}{2} C)^2$, and therefore,

$4AC.CE : 2AC.DE :: R^2 : (\sin. \frac{1}{2} C)^2$. And it was proved

that $2AC.ED = (AB + AE)(AB - AE)$, therefore

$4AC.CE : (AB + AE)(AB - AE) :: R^2 : (\sin. \frac{1}{2} C)^2$.

Case 2. Let C be an obtuse angle, so that the perpendicular DB falls without the triangle. Then, because $AC^2 + CB^2 + 2AC \cdot CD = AB^2$ (12. 2.), or $AC^2 + CE^2 + 2AC \cdot CD = AB^2$; and because $AC^2 + CE^2 = 2AC \cdot CE + AE^2$, $2AC \cdot CE + 2AC \cdot CD + AE^2 = AB^2$. Now, $2AC \cdot CE + 2AC \cdot CD = 2AC \cdot ED$, therefore $2AC \cdot ED + AE^2 = AB^2$. Therefore, it will be demonstrated as before, that $4AC \cdot CE : (AB + AE)(AB - AE) :: R^2 : (\sin. \frac{1}{2} C)^2$.



Now, $4AC \cdot CE$, or $4AC \cdot CB$ is four times the rectangle under the sides, and AE being the difference of the sides, $(AB + AE)(AB - AE)$ is the rectangle which has for one of its sides the base increased by the difference of the two sides of the triangle, and for its other side, the base diminished by the same difference. Therefore, in any triangle, &c.
Q. E. D.

Cor. Because BDC is a right angle, $BC : CD :: R : \sin. CBD$; now, $\sin. CBD = \cos. C$, therefore, $BC : CD :: R : \cos. C$, and $AC \cdot CB : 2AC \cdot CD :: R : \cos. C$. But, (13. 2.) $AC \cdot CD = AC^2 + CB^2 - AB^2$, therefore, $AC \cdot CB : AC^2 + CB^2 - AB^2 :: R : \cos. C$; that is, twice the rectangle $AC \cdot CB$, is to the excess of the squares of AC and CB above the square of AB , as radius to the co-sine of the included angle ACB .

P R O P.

P R O P. VIII.

FOUR times the rectangle contained by any two sides of a triangle, is to the rectangle contained by two straight lines, of which one is the sum of those sides increased by the base of the triangle, and the other the sum of the same sides diminished by the base, as the square of the radius to the square of the co-sine of half the angle included between the two sides of the triangle.

The same construction being made as in the last proposition, it is plain, that AF is the sum, and AE the difference of the sides AC and CB, and also, that the angle FEB is half of the angle FCB. Now, if ACB be an acute angle $AC^2 + CB^2$, or $AC^2 + CF^2 = AB^2 + 2AC.CD$; add $2AC.CF$ to both, and $AC^2 + CF^2 + 2AC.CF$, that is, $AF^2 = AB^2 + 2AC.CD + 2AC.CF$; But $2AC.CD + 2AC.CF = 2AC.DF$, because $CD + CF = DF$, and therefore, $AF^2 = AB^2 + 2AC.DF$; and $2AC.DF = AF^2 - AB^2 = (AF + AB)(AF - AB)$, (2. 5.).

Again, $AC : CF :: 2AC.DF : 2CF.DF$, and $4AC.CF : 4CF^2 :: 2AC.DF : 2CF.DF$; and alternately $4AC.CF : 2AC.DF :: 4CF^2 : 2CF.DF$; now, $4CF^2 = EF^2$, and $2CF.DF = EF.DF = FB^2$, therefore, $4AC.CF : 2AC.DF :: EF^2 : FB^2$. But $EF : FB :: R : \sin. FEB$, and $\sin. FEB = \cos. EFB$, because EFB is the complement of FEB, so that

$\sin. FEB = \cos. \frac{1}{2} ECB$; therefore, $EF : FB :: R : \cos. \frac{1}{2} ECB$

and $EF^2 : FB^2 :: R^2 : (\cos. \frac{1}{2} ECB)^2$ Therefore,

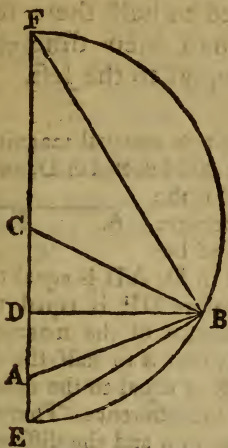
$4AC.CF : 2AC.DF :: R^2 : (\cos. \frac{1}{2} ECB)^2$, or

$4AC.CB : (AF + AB)(AF - AB) :: R^2 : (\cos. \frac{1}{2} ECB)^2$.

In the same way, when the angle C is obtuse, it will be demonstrated, that $4AC.CB : (AF + AB)(AF - AB) :: R^2 : (\cos. \frac{1}{2} ECB)^2$. Now, $4AC.CB$ is four times the rectangle under the sides, and AF being the sum of the sides, $(AF + AB)(AF - AB)$ is the rectangle which has one of

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des equal to the sum of the sides of the triangle increased by the base, and its other side equal to the sum of the sides of the triangle diminished by the base. Q. E. D.



COR. Because $4AC \cdot CB : (AF + AB)(AF - AB) :: R^2 :$

$\text{cof. } \frac{1}{2} ECB)^2$, and also (prop. 7.) $4AC \cdot CB :$

$(AB + AE)(AB - AE) :: R^2 : (\text{fin. } \frac{1}{2} ECB)^2$; therefore,

æquo, $(AF + AB)(AF - AB) : (AB + AE)(AB - AE) ::$

$\text{cof. } \frac{1}{2} ECB)^2 : (\text{fin. } \frac{1}{2} ECB)^2$.

ut, $\text{cof. } \frac{1}{2} ECB : \text{fin. } \frac{1}{2} ECB :: R : \tan. \frac{1}{2} ECB$;

therefore, $(AF + AB)(AF - AB) : (AB + AE)(AB - AE) ::$

$^2 : (\tan. \frac{1}{2} ECB)^2$; that is, the rectangle contained by two

straight lines, of which one is the sum of the three sides of a triangle, and the other the sum of two of the sides of the triangle diminished by the third, is to the rectangle contained by two straight lines, of which one is the third side of the triangle increased by the difference of the other two, and the other the third side diminished by the difference of the other two, as the square of the radius to the square of the tangent of half the angle included between the two sides of the triangle.

L E M M A II.

IF there be two unequal magnitudes, half the difference added to half their sum is equal to the greater; and half their difference taken from half their sum is equal to the less.

Let AB and BC be two unequal magnitudes, of which A is the greater; suppose AC bisected in D , and AE equal to BC . It is manifest, that AC is the sum, and EB the difference $\begin{array}{ccccccc} & & A & & E & & D & & B \end{array}$ of the magnitudes. And because AC is bisected in D , AD is equal to DC ; but AE also equal to BC , therefore DE is equal to DB , and DE DB is half the difference of the magnitudes. But AB equal to BD and DA , that is to half the difference added half the sum; and BC is equal to the excess of DC , half the sum, above DB , half the difference. Therefore, &c. Q. E.

COR. Hence, if the sum and the difference of two magnitudes be given, the magnitudes themselves may be found.

SOLUTION of the Cases of Right Angled Triangles.

P R O B L E M.

IN a right angled triangle of the three sides and three angles, any two being given, besides the right angle, and one of those two being a side, is required to find the other three.

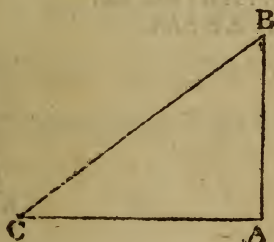
The reason of the limitation in this Prop. to those cases which at least one side is given, is manifest; for, if the angle only be given, the sides cannot be found, but only their ratios, which are the same with the ratios of the sines of the opposite angles.

It is evident, that when one of the acute angles of a right angled triangle is given, the other is given, being the complement of the former to a right angle; and also that the sine of any of the acute angles is the co-sine of the other.

As this general problem admits of several cases, the solutions or rules for calculation, which all depend on the first proposition, may be conveniently exhibited in the form of a table.

When two sides of a right angled triangle are given in numbers, the third may be found, either trigonometrically, as in the table, or by the 47th of the 1st of the Elem. Thus, because $BC^2 = AB^2 + AC^2$, if AB and AC are given, BC is found, for it is the square root of $AB^2 + AC^2$. In the same manner, if AB and BC are given, AC is found, being the square root of $BC^2 - AB^2$.

GIVEN.	SOUGHT.	SOLUTION.	
CB and B, the hypotenuse and an angle.	AC:	$R : \sin. B :: CB : AC.$	1
	AB.	$R : \cos. B :: CB : AB.$	1
AC and C; a side, and one of the acute angles.	BC.	$\sin. B : R :: AC : BC.$	2
	AB.	$R : \tan. C :: AC : AB.$	3
CB and BA, the hypotenuse and a side.	C.	$CB : BA :: R : \sin. C.$	4
	AC.	$R : \cos. C :: CB : AC.$	1
AC and AB; the two sides.	C.	$AC : AB :: R : \tan. C.$	5
	CB.	$\cos. C : R :: AC : CB.$	2



S O L U-

SOLUTION of the Cases of Oblique Angled Triangles.

PROBLEM.

IN any triangle of the three sides and three angles any three being given, and one of the three being a side, it is required to find the other three.

The reason of the restriction, that one of the things given must be a side, is evident; because, if the angles only are given, the sides of the triangle cannot be found, but only the ratios, which are the same with the ratios of the sines of opposite angles.

In this problem there are four cases, in each of which solution depends on some of the foregoing propositions.

CASE I.

Two angles A and B, and one side AB, of a triangle ABC being given, to find the other sides.

SOLUTION.

Because the angles A and B are given, C is also given, being the supplement of A+B; and, (Prop. 2.)

$\text{Sin. C} : \text{fin. A} :: \text{AB} : \text{BC}$, also

$\text{Sin. C} : \text{fin. B} :: \text{AB} : \text{AC}$.



CASE

C A S E II.

Two sides AB and AC, and the angle B opposite to one of them being given, to find the other angles A and C, and also the other side BC.

SOLUTION.

$AC : AB :: \sin. B : \sin. C$. Also, $A = 180^\circ - B - C$; and $B : \sin. A :: AC : CB$, by case I.

In this case, the angle C may have two values; for its sine being found by the proportion above, the angle belonging to that sine, may either be that which is found in the tables, or may be the supplement of it, (Cor. def. 4.). But this ambiguity is removed when AC is greater than AB, for the angle ACB cannot in that case be obtuse, (otherwise AB would be greater than AC), and therefore the greatest of the two values of C is necessarily excluded. When AC is less than AB, C remains ambiguous, and there are two triangles that have the sides AB, AC, and the angle B the same in both, but the angle opposite to AB in the one, is the complement of the angle opposite to AB in the other.

C A S E III.

Two sides AB and AC, and the angle A between them being given, to find the other angles B and C, and also the side BC.

SOLUTION.

$AB + AC : AB - AC :: \tan. \frac{1}{2} (B + C) : \tan. \frac{1}{2} (B - C)$.

Hence, by the 2d Lemma, B and C are found.

Otherwise,

Find an angle E , such that $AB : BC :: R : \tan. E$; then $R : \tan. (45^\circ - E) :: \tan. (B+C) : \tan. (B-C)$. When B and C are found, as before.

To find BC .

Having found B , $\sin. B : \sin. A :: AC : BC$.

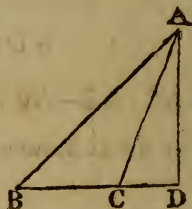
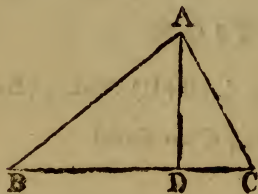
C A S E IV.

The three sides AB , BC , AC , being given, to find the angles A , B , C .

SOLUTION

Take F such that $BC : BA + AC : BA - AC : F$, then F is either the sum or the difference of BD , DC , the segments of the base, (Prop. 5.) If F be greater than BC , F is the sum and BC the difference of BD , DC ; but, if F be less than BC , BC is the sum, and F the difference of BD and DC . In either case, the sum of BD and DC , and their difference being given, BD and DC are found.

Then, (prop. 1.) $CA : CD :: R : \cos. C$; and $BA : BD :: R : \cos. B$; wherefore C and B are given and consequently A .



Otherw

Otherwise,

Let D be the difference of the sides AB, AC, then, (prop. 7.)

$$AB.AC : (BC+D)(BC-D) :: R^2 : (\sin. \frac{1}{2} BAC)^2, \text{ or}$$

$$\sqrt{AB.AC} : \sqrt{(BC+D)(BC-D)} :: R : \sin. \frac{1}{2} BAC.$$

Otherwise,

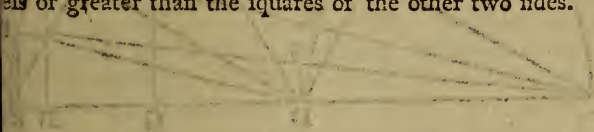
Let S be the sum of the sides BA and AC. Then (prop. 8.)

$$AB.AC : (S+BC)(S-BC) :: R^2 : (\sin. \frac{1}{2} BAC)^2 \text{ or}$$

$$\sqrt{AB.AC} : \sqrt{(S+BC)(S-BC)} :: R. \cos. \frac{1}{2} BAC.$$

Other solutions of this case might also be given from the corollaries to the 7th and 8th propositions, but the above are sufficient.

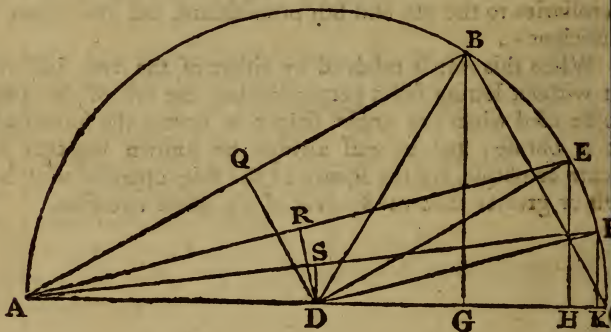
When this case is resolved by either of the two last rules, without letting fall a perpendicular, the first of the two is to be used when the angle sought is acute; the second when it is obtuse; and it will always be known whether it is acute or obtuse, by the square of the side opposite to it being less or greater than the squares of the other two sides.



SCHOLIUM.

In all the calculations performed by the preceding rules, tables of sines and tangents are necessarily employed, the construction of which it is therefore proper to explain. The tables usually contain the sines, &c. to every minute of the quadrant from $1'$ to 90° , and the first thing required to be done is to compute the sine of $1'$.

1. In the circle ABC the radius AD being unity, let BC be any arch, and CE its half; then, if DG the co-sine of BC be known, the co-sine of CE may be found. From E draw EH perpendicular to DC , so that EH may be the sine, and DH the co-sine of the arch CE . Join AB and BC , and let DQ be perpendicular to AB , then AB is bisected in Q . Now, the



triangles DHE , AQD are equal, the angle DAQ being equal to the angle EDH , (because each of them is the half of the angle CDB), and the angles DHE , AQD being right angles, and also the side AD equal to the side DE ; therefore DH is equal to AQ . And because ABC is a right angle, the rectangle CA , AG is equal to the square of AB , (8. 6. th

that is, to four times the square of AQ or of DH ; and therefore, half the rectangle DA, AH is equal to the square of DH , that is, half the rectangle contained by the radius, and the sum of the radius and the co-sine of an arch, is equal to the square of the co-sine of half that arch. And hence, this arithmetical rule, to the co-sine of any arch add the radius or 1, the square root of half the sum is the co-sine of half the arch.

2. If therefore the straight line CB be equal to the radius, the arch CB will be 60° , and DG will be equal to half the radius as was formerly shewn (8. 8.), and therefore DH the co-sine of CE or of 30° , will be found by this proposition. And if CE be bisected in F , if to DH 1 be added, and the sum divided by 2, its square root, will be equal to DK , the co-sine of 15° . In the same manner is found the co-sine of $7^\circ 30'$, of $3^\circ 45'$, and so on till, after twelve bisections, the co-sine of $52''.44'''.03'''.45''''$ is found. Now, from the co-sine, the sine of the same arch is found; for, from the square of the radius, that is, from 1, take away the square of the co-sine, the remainder is the square of the sine, and therefore its square root is the sine itself. Thus, the sine of $52''.44'''.03'''.45''''$ is found.

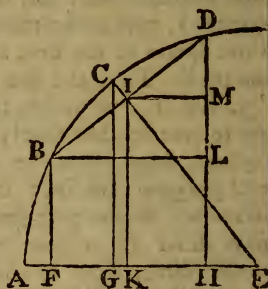
3. But it is manifest, that the sines of very small arches are to one another nearly as the arches themselves. For it has been shewn, that the number of the sides of an equilateral polygon inscribed in a circle may be so great, that the perimeter of the polygon and the circumference of the circle may differ by a line less than any given line, or which is the same, may be nearly to one another in the ratio of equality. Therefore their like parts will also be nearly in the ratio of equality, so that the side of the polygon will be to the arch which it subtends nearly in the ratio of equality; and therefore, half the side of the polygon to half the arch subtended by it, that is to say, the sine of any very small arch will be to the arch itself, nearly in the ratio of equality. Therefore, if two arches are both very small, the first will be to the second as the sine of the first to the sine of the second. Hence, from the sine of $52''.44'''.03'''.45''''$ being found, the sine of $1'$ becomes known; for, as $52''.44'''.03'''.45''''$ to $1'$, so the sine of the former arch to the sine of the latter. Thus, the sine of $1'$ is found = 0.0002908882.

4. The sine of $1'$ being thus found, the sines of $2'$, of $3'$ of any number of minutes are found by the following proposition.

T H E O R E M.

Let AB, AC, AD be three such arches, that BC the difference of the first and second is equal to CD the difference of the second and third; the radius is to the co-sine of the common difference BC as the sine of AC the middle arch, to half the sum of the sines of AB and AD the extreme arches.

Draw CE to the centre; let $BF, CG,$ and DH perpendicular to AE , be the sines of the arches AB, AC, AD . Join BD , and let it meet CE in I ; draw IK perpendicular to AE , also BL and IM perpendicular to DH . Then, because the arch BD is bisected in C , EC will be at right angles to BD , and will bisect it in I , and BI will be the sine, and EI the co-sine of BC or CD . And, since BD is bisected in I and IM is parallel to BL , (2. 6.) LD is also bisected in M . Now BF is equal to HL , therefore, $BF + DH = DH + HL = DL + 2LH = 2LM + 2LH = 2MH$, or $2KI$; and therefore, IK is half the sum of BF and DH . But because the triangles CGE, IKE are equiangular, $CE : EI :: CG : IK$, and it has been shewn that $EI = \text{cof. } BC$, and $IK = \frac{1}{2}(BF + DH)$; therefore $R : \text{cof. } BC :: \text{fin. } AC : \frac{1}{2}(\text{fin. } AB + \text{fin. } AD)$. Q. E. D.



COR. Hence, if the point B coincides with A,

$R : \text{cof. } BC :: \text{fin. } BC : \frac{1}{2} \text{ fin. } BD$, or the radius is to the co-sine of the half of any arch, as the sine of half the arch is to half the sine of the whole arch.

Therefore, the sine of $1'$ being given, and also its co-sine, because by this Cor. $R : \text{cof. } 1' :: \text{fin. } 1' : \frac{1}{2} \text{ fin. } 2'$; and so the sine of $2'$ is given. To find the sine of $3'$; by the preceding

Theor. $R : \text{cof. } 1' :: \text{fin. } 2' : \frac{1}{2} (\text{fin. } 1' + \text{fin. } 3')$; therefore, $\text{fin. } 1' + \text{fin. } 3'$ is given, and $\text{fin. } 1'$ being already known, $\text{fin. } 3'$ is found.

In like manner, for the sine of $4'$,

$R : \text{cof. } 1' :: \text{fin. } 3' : \frac{1}{2} (\text{fin. } 2' + \text{fin. } 4')$; and so on for any number of minutes; and as the ratio of R to the co-sine of $1'$ remains always the same, the calculation by means of this Theorem is very easy.

It may be convenient to have the same rule otherwise expressed, let a, b, c , be three arches that differ by $1'$, then,

$R : \text{cof. } 1' :: \text{fin. } b : \frac{1}{2} (\text{fin. } a + \text{fin. } c)$, and if $R = 1$,

$\frac{1}{2} (\text{fin. } a + \text{fin. } c) = \text{cof. } 1' \times \text{fin. } b$, and consequently,

$\text{fin. } c = 2 \text{ cof. } 1' \times \text{fin. } b - \text{fin. } a$.

In this manner, the table of sines is computed. Then, because the co-sine of any arch is to the sine as the radius to the tangent of the arch, the table of tangents may be calculated from this proportion. But it must be observed, that when the tangents of the arches under 45° are known, the tangents of those above 45° may be more easily found by another rule.

For the tangents of the arches above 45° being the co-tangents of the arches under 45° , and the radius being a mean proportional between the tangent and co-tangent of any arch, (1. Cor. def. 9.) if the difference between any arch and 45° be called d , $\tan. (45^\circ - d) : 1 :: 1 : \tan. (45^\circ + d)$.

Lastly, The secants are calculated by help of Cor. 2. Def. 9. where it is shewn, that the radius is a mean proportional between the co-sine and the secant of any arch. If therefore, a be any arch, $\text{cof. } a : 1 :: 1 : \text{sec. } a$.

The

5. The preceding Theorem is one of four, which, when arithmetically expressed, are frequently used in the application of trigonometry to the solution of the more difficult problems.

1^{mo}, If in the last Theorem, the arch $AC=A$, $BC=B$, and $EC=1$; then $AD=A+B$, and $AB=A-B$; and by what has just been demonstrated,

$$1 : \text{cof. } B :: \text{fin. } A : \frac{1}{2} \text{fin. } (A+B) + \frac{1}{2} \text{fin. } (A-B),$$

and therefore,

$$\text{fin. } A \times \text{cof. } B = \frac{1}{2} \text{fin. } (A+B) + \frac{1}{2} \text{fin. } (A-B).$$

2^{do}, Because BF IK , DH are parallel, the straight lines BD and FH are cut proportionally, and therefore FH , the difference of the straight lines FE and HE , is bisected in K ; and therefore, as was shewn in the last Theorem, KE is half the sum of FE and HE , that is, of the co-sines of the arches AB and AD . But because of the similar triangles EGC , EKI , $EC : EI :: GE : EK$; now, GE is the co-sine of AC , therefore,

$$R : \text{cof. } BC :: \text{cof. } AC : \frac{1}{2} \text{cof. } AD + \frac{1}{2} \text{cof. } AB,$$

$$\text{or } 1 : \text{cof. } B :: \text{cof. } A : \frac{1}{2} \text{cof. } (A+B) + \frac{1}{2} \text{cof. } (A-B);$$

and therefore,

$$\text{cof. } A \times \text{cof. } B = \frac{1}{2} \text{cof. } (A+B) + \frac{1}{2} \text{cof. } (A-B).$$

3^{tio}, Again, the triangles IDM , CEG are equiangular, for the angles KIM , EID are equal, being each of them right angles, and therefore, taking away the angle EIM , the angle DIM is equal to the angle EIK , that is, to the angle ECG ; and the angles DMI , CGE are also equal, being both right angles, and therefore, the triangles IDM , CEG have the sides about their equal angles proportionals, and consequently, $EC : CG :: DI : IM$; now, IM is half the difference of the co-sines FE and EH , therefore

$$R : \text{fin. } AC : \text{fin. } BC :: \frac{1}{2} \text{cof. } AB - \frac{1}{2} \text{cof. } AD,$$

$$\text{or } 1 : \text{fin. } A : \text{fin. } B :: \frac{1}{2} \text{cof. } (A-B) - \frac{1}{2} \text{cof. } (A+B);$$

and also,

$$\text{fin. } A \times \text{fin. } B = \frac{1}{2} \text{cof. } (A-B) - \frac{1}{2} \text{cof. } (A+B).$$

4^{to}, Lastly, In the same triangles ECG, DIM,
 $EC : EG :: ID : DM$; now, DM is half the difference of the
 lines DH and BF, therefore,

$$R : \text{cof. } AC :: \text{fin. } BC : \frac{1}{2} \text{ fin. } AD - \frac{1}{2} \text{ fin. } AB.$$

$$\text{or } 1 : \text{cof. } A :: \text{fin. } B : \frac{1}{2} \text{ fin. } (A+B) - \frac{1}{2} \text{ fin. } (A-B) ;$$

and therefore,

$$\text{cof. } A \times \text{fin. } B = \frac{1}{2} \text{ fin. } (A+B) - \frac{1}{2} \text{ fin. } (A-B).$$

6. If therefore A and B be any two arches whatsoever,
 the radius being supposed 1 ;

$$\text{I. fin. } A \times \text{cof. } B = \frac{1}{2} \text{ fin. } (A+B) + \frac{1}{2} \text{ fin. } (A-B).$$

$$\text{II. cof. } A \times \text{cof. } B = \frac{1}{2} \text{ cof. } (A-B) + \frac{1}{2} \text{ cof. } (A+B).$$

$$\text{III. fin. } A \times \text{fin. } B = \frac{1}{2} \text{ cof. } (A-B) - \frac{1}{2} \text{ cof. } (A+B).$$

$$\text{IV. cof. } A \times \text{fin. } B = \frac{1}{2} \text{ fin. } (A+B) - \frac{1}{2} \text{ fin. } (A-B).$$

From these four Theorems are also deduced other four.

For, by adding the first and fourth together,

$$\text{fin. } A \times \text{cof. } B + \text{cof. } A \times \text{fin. } B = \text{fin. } (A+B).$$

Also, by taking the fourth from the first,

$$\text{fin. } A \times \text{cof. } B - \text{cof. } A \times \text{fin. } B = \text{fin. } (A-B).$$

Again, adding the second and third,

$$\text{cof. } A \times \text{cof. } B + \text{fin. } A \times \text{fin. } B = \text{cof. } (A-B) ;$$

And, lastly, subtracting the third from the second,

$$\text{cof. } A \times \text{cof. } B - \text{fin. } A \times \text{fin. } B = \text{cof. } (A+B).$$

E L E.

ELEMENTS

OF

SPHERICAL TRIGONOMETRY

DEFINITIONS.

I.

A Great circle of the sphere is any circle on the superficies of the sphere of which the plane passes through the centre of the sphere, and of which the centre therefore is the same with the centre of the sphere.

II.

The pole of a great circle of the sphere is a point in the superficies of the sphere, from which all straight lines drawn to the circumference of the circle are equal.

III.

A spherical angle is that which on the superficies of a sphere is contained by two arches of great circles, and is the same with the inclination of the planes of these great circles.

IV.

A spherical triangle is a figure upon the superficies of a sphere comprehended by three arches of three great circles, each of which is less than a semicircle.

P R O P.

P R O P. I.

ANY two great circles of a sphere bisect one another.

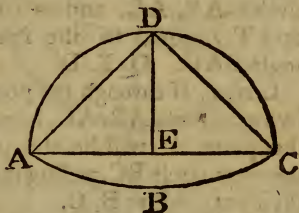
For, as they have the same centre, their common section is diameter of both, and therefore bisects both.

P R O P. II.

THE arch of a great circle between the pole and the circumference of another great circle is a quadrant.

Let ABC be a great circle, and D its pole; if a great circle CD pass through D, and meet ABC in C, the arch DC is quadrant.

Let the great circle CD meet ABC again in A, and let AC be the common section of the great circles, which will pass through E the centre of the sphere: Join DE, DA, DC: by def. 2. DA, DC are equal, and AE, EC are also equal, and DE is common; therefore (8. 1.) the angles DEA, DEC are equal; wherefore the arches DA, DC are equal, and consequently each of them is a quadrant. Q. E. D.



COR. The circle ABC has two poles, one on each side of its plane, which are the extremities of a diameter of the sphere perpendicular to the plane ABC; and no other point but these two can be a pole of the circle ABC.

P R O P.

P R O P. III.

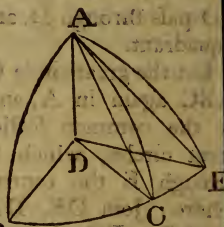
IF the pole of a great circle be the same with the intersection of other two great circles; the arc of the first mentioned circle intercepted between the other two, is the measure of the spherical angle which the same two circles make with one another.

Let the great circles BA, CA on the superficies of a sphere of which the centre is D, intersect one another in A, and let BC be an arch of another great circle, of which the pole is A; BC is the measure of the spherical angle BAC.

Join AD, DB, DC; since A is the pole of BC, AB, AC are quadrants, and the angles ADB, ADC are right angles; therefore (4. def. 7.) the angle CDB is the inclination of the planes of the circles AB, AC, and is (def. 3. Sp. T.) equal to the spherical angle BAC. Q. E. D.

COR. 1. If through the point A, two quadrants AB, AC, be drawn, the point A will be the pole of the great circle BC, passing through their extremities B, C.

Join AC, and draw AE, a straight line to any other point E in EC; join DE: Since AC, AB are quadrants, the angle ADB, ADC are right angles, and AD will be perpendicular to the plane of BC: Therefore the angle ADE is right angle, and AD, DC are equal to AD, DE each to each therefore AE, AC are equal, and A is the pole of BC, by def. 2.



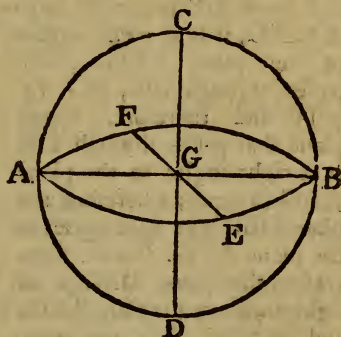
P R O P. IV.

If there be two great circles of a sphere, of which the first passes through the poles of the second, the second also passes through the poles of the first.

Let ACBD and AEBF be two circles, the one of which ACBD passes through G and D, the poles of the other AEBF; the circle AEBF passes through the poles of the circle ACBD.

Let G be the centre of the sphere; join CG, GD, which will be in the same straight line: Draw AGB the diameter, which is the common section of the circles ACBD, AEBF; and in the plane of the circle AEBF draw from the point G the straight line EGF perpendicular to AB.

Then, because C and D are the poles of the circle AEBF, the arch intercepted between either of them and any point in the circumference AEBF, is a quadrant (Prop. 3.), and therefore the straight line CD is perpendicular to the plane of the circle AEBF; wherefore also the plane of the circle ACBD, which passes through CD, is perpendicular (17. 7.) to the plane AEBF. And since GE in the plane AEBF, is perpendicular to AB, the common section of the two planes, it is perpendicular to the plane (2. def. 7.) ACBD. The arch of a great circle, therefore, intercepted between the point E and any point of the circumference ACBD is a quadrant, and therefore, (Prop. 3.) the point E is the pole of the circle ACBD. For the same reason, F is the other pole of the circle ACBD; and E and F are in the circumference of the circle AEBF. Q. E. D.



P R O P. V.

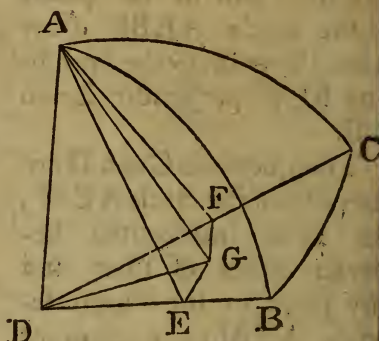
IN isosceles spherical triangles the angles at the base are equal.

Let ABC be a spherical triangle, having the side AB equal to the side AC ; the spherical angles ABC and ACB are equal.

Let D be the centre of the sphere; join DB , DC , DA and from A on the straight lines DB , DC , draw the perpendiculars AE , AF ; and from the points E and F draw in the plane DBC the straight lines EG , FG perpendicular to DE and DC , meeting one another in G : Join AG .

Because DE is at right angles to each of the straight lines AE , EG , it is at right angles to the plane AEG , which passes through AE , EG , (4. 7.); and therefore, every plane that passes through DE is at right angles to the plane AEG (17. 7.); wherefore, the plane DBC is at right angles to the plane AEG . For the same reason, the plane DBC is at right angles to the plane AFG ; and therefore AG , the common section of the planes AEG , AFG is at right angles (18. 7.) to the plane DBC , and the angles AGE , AGF are consequently right angles.

But, since the arch AB is equal to the arch AC , the angle ADB is equal to the angle ADC . Therefore the triangles ADE , ADF , have the angles EDA , FDA equal, as also the angles AED , AFD , which are right angles; and they have the side AD common, therefore the other sides are equal, viz AE to AF , (26. 1.) and DE to DF . Again, because the angles AGE , AGF are right angles, the squares on AG and GE are equal to the square of AE ; and the squares of AG and GF to the square of AF . But the squares of AE and AF are equal, therefore the squares of AG and GE are equal to the squares of AG and GF , and taking away the common square



square of AG , the remaining squares of GE and GF are equal, and GE is therefore equal to GF . Wherefore, in the triangles AFG , AEG , the side GF is equal to the side GE , and AF has been proved to be equal to AE , and the base AG is common, therefore the angle AFG is equal to the angle AEG (8. 1.). But the angle AFG is the angle which the plane ADC makes with the plane DBC (4. def. 7.), because FA and FG , which are drawn in these planes, are at right angles to DF , the common section of the planes. The angle AFG (3. def. Sp. T.) is therefore equal to the spherical angle ACB ; and for the same reason, the angle AEG is equal to the spherical angle ABC . But the angles AFG , AEG are equal. Therefore the spherical angles ACB , ABC are also equal. Q. E. D.

P R O P. VI.

IF the angles at the base of a spherical triangle be equal, the triangle is isosceles.

Let ABC be a spherical triangle having the angles ABC , ACB equal to one another; the sides AC and AB are also equal.

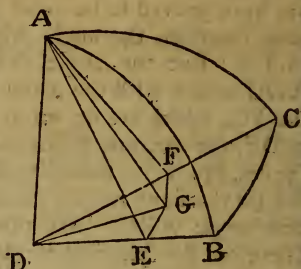
Let D be the centre of the sphere; join DB , DC , DA , and from A on the straight lines DB , DC , draw the perpendiculars AE , AF ; and from the points E and F , draw in the plane DBC the straight lines EG , FG perpendicular to DB and DC , meeting one another in G ; join AG .

Then, it may be proved as was done in the last Prop. that AG is at right angles to the plane BCD , and that therefore the angles AGF , AGE are right angles, and also that the angles AFG , AEG are equal to the angles which the planes DAC , DAB make with the plane DBC . But because the spherical angles ACB , ABC are equal, the angles which the planes DAC , DAB make with the plane DBC are equal, (3. def. Sph. Tr.) and therefore the angles AFG , AEG are also equal. The triangles AGE , AGF have therefore two angles of the one equal to two angles of the other, and they have also the side AG common, wherefore they are equal, and the side AF is equal to the side AE .

Y 2

Again,

Again, because the triangles ADF , ADE are right angled at F and E , the squares of DF and FA are equal to the square of DA , that is, to the squares of DE and EA ; now, the square of AF is equal to the square of AE , therefore the square of DF is equal to the square of DE , and the side DF to the side DE . Therefore in the triangles DAF , DAE , because DF is equal to DE , and DA common, and also AF equal to AE , the angle ADF is equal to the angle ADE ; therefore also, the arches AC and AB , which are the measures of the angles ADF and ADE are equal to one another; and the triangle ABC is isosceles. Q. E. D.



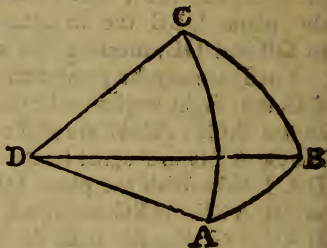
P R O P. VII.

ANY two sides of a spherical triangle are greater than the third.

Let ABC be a spherical triangle, any two sides AB , BC are greater than the third side AC .

Let D be the centre of the sphere; join DA , DB , DC .

The solid angle at D is contained by three plane angles ADB , ADC , BDC ; (10. 7.) any two of which ADB , BDC are greater than the third ADC , that is, any two sides AB , BC of the spherical triangle ABC , are greater than the third AC . Q. E. D.



P R O P. VIII.

THE three sides of a spherical triangle are less than a circle.

Let ABC be a spherical triangle as before, the three sides AB , BC , AC are less than a circle.

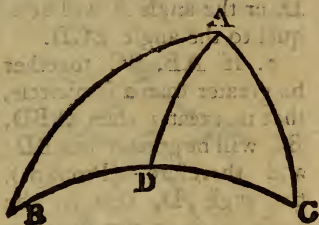
Let D be the centre of the sphere: The solid angle at D is contained by three plain angles BDA , BDC , ADC , which together are less than four right angles, (20. 7.) therefore the sides AB , BC , AC together, will be less than four quadrants, that is less than a circle. Q. E. D.

P R O P. IX.

IN a spherical triangle the greater angle is opposite to the greater side; and conversely.

Let ABC be a spherical triangle, the greater angle A is opposed to the greater side BC .

Let the angle BAD be made equal to the angle B , and then BD , DA will be equal, (Prop. 6.) and therefore AD , DC are equal to BC ; but AD , DC are greater than AC , (Prop. 7.), therefore BC is greater than AC , that is, the greater angle A is opposite to the greater side BC . The converse is demonstrated as Prop. 19: 1. Elem. Q. E. D.



P R O P. X.

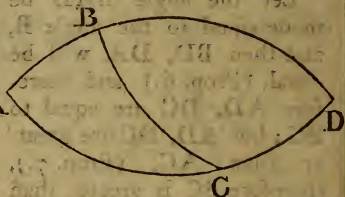
IN a spherical triangle, according as the sum of two of the sides is greater than a semicircle, equal to it, or less, the interior angle at the base is greater than the exterior and opposite angle at the base, equal to it, or less; and the sum of the two interior angles at the base greater than two right angles, equal to two right angles, or less than two right angles.

Let ABC be a spherical triangle, of which the sides are AB and BC ; produce the side AB and the base AC till they meet again in D ; then, the arch ABD is a semicircle, and the spherical angles at A and D are equal, because each of them is the inclination of the circle ABD to the circle ACD .

1. If AB, BC be equal to a semicircle, that is, to AD , BC will be equal to BD , and therefore (Prop. 5.) the angle D , or the angle A will be equal to the angle BCD .

2. If AB, BC together be greater than a semicircle, that is greater than ABD , BC will be greater than BD ; and therefore (Prop. 9.), the angle D , that is, the angle A , is greater than the angle BCD .

3. In the same manner it is shewn, if AB, BC together be less than a semicircle, that the angle A is less than the angle BCD . And since the angles BCD, BCA are equal to two right angles, if the angle A be greater than BCD , A and ACB together will be greater than two right angles. If A be equal to BCD , A and ACB together will be equal to two right



right angles; and if A be less than BCD , A and ACB will be less than two right angles. Q. E. D.

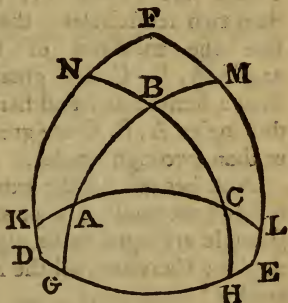
P R O P. XI.

If the angular points of a spherical triangle be made the poles of three great circles, these three circles by their intersections will form a triangle which is said to be supplemental to the former; and the two triangles are such, that the sides of the one are the supplements of the arches which measure the angles of the other.

Let ABC be a spherical triangle; and from the points A , B , and C as poles, let the great circles FE , ED , DF be described, intersecting one another in F , D and E ; the sides of the triangle FED are the supplements of the measures of the angles A , B , C , viz. FE of the angle BAC , DE of the angle ABC , and DF of the angle ACB : And again, AC is the supplement of the angle DFE , AB of the angle FED , and BC of the angle EDF .

Let AB produced meet DE , EF in G , M ; let AC meet DE , FE in K , L ; and let BC meet DE , DF in N , H .

Since A is the pole of FE , and the circle AC passes through A , F will pass through the pole of AC (Prop. 4.), and since AC passes through C , the pole of ED , FD will pass through the pole of AC ; therefore the pole of AC is in the point F , in which the arches DF , EF intersect each other. In the same manner, D is the pole of BC , and E the pole of AB .



And since F , E are the poles of AL , AM , the arches FL and EM are quadrants, and FL , EM together, that is, FE and L together are equal to a semicircle. But since A is the pole

pole of ML, ML is the measure of the angle BAC, (Prop. 3) consequently FE is the supplement of the measure of the angle BAC. In the same manner, ED, DF are the supplements of the measures of the angles ABC, BCA.

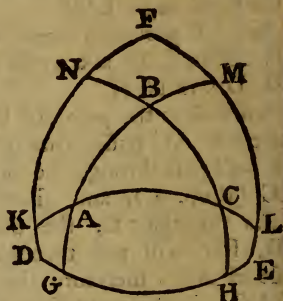
Since likewise CN, BH are quadrants, CN, BH together that is, NH, BC together, are equal to a semicircle; and since D is the pole of NH, NH is the measure of the angle FDE therefore the measure of the angle FDE is the supplement of the side BC. In the same manner, it is shewn that the measures of the angles DEF, EFD are the supplements of the sides AB, AC, in the triangle ABC. Q. E. D.

P R O P. XII.

THE three angles of a spherical triangle are greater than two right angles, and less than six right angles.

The measures of the angles A, B, C, in the triangle ABC together with the three sides of the supplemental triangle DEF, are (Prop. 11.) equal to three semicircles; but the three sides of the triangle FDE, are (Prop. 8.) less than two semicircles; therefore the measures of the angles A, B, C are greater than a semicircle; and hence the angles A, B, C are greater than two right angles.

And because all the external and internal angles of any triangle are equal to six right angles; therefore, all the internal angles are less than six right angles.

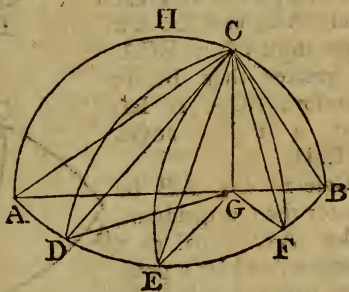


P R O P. XIII.

F to the circumference of a great circle, from a point which is not the pole of it, arches of great circles be drawn; the greatest of these arches is that which passes through the pole of the first mentioned circle, and the supplement of it is the least; and of the others, that which is nearer to the greatest is greater than that which is more remote.

Let ADB be the circumference of a great circle, of which the pole is H, and let C be any other point; through C and H let the semicircle ACB be drawn meeting the circle ADB in A and B; and let the arches CD, CE, CF also be described. From C draw CG perpendicular to AB, and then, because the circle AHCB which passes through H, the pole of the circle ADB, is at right angles to ADB, CG is perpendicular to the plane ADB. Join GD, GE, GF, CD, CE, CF, CA, CB.

Of all the straight lines drawn from G to the circumference ADB, GA is the greatest, and GB the least (7. 3.); and GD which is nearer to GA is greater than GE, which is more remote. But the triangles CGA, CGD are right angled at G, and they have the common side CG; therefore the squares of CG, GA together, that is, the square of CA, is greater than the squares of CG, GD together, that is, than the square of CD; therefore CA is greater than CD, and the arch CA greater than the arch CD. In the same manner, since GD is greater than GE, and GE than GF, it is shewn that CD is greater than CE, and CE than CF, and consequently, the arch CD greater than the arch CE, and the arch CE greater than the arch CF. Also, because AG is the greatest, and GB the least of all the lines drawn from G, CA is the greatest, and CB the least of all the lines drawn from C, and therefore the arch CA is the greatest, and CB, its supplement, the least of all the arches drawn through C. Q. E. D.



P R O P.

P R O P. XIV.

IN a right angled spherical triangle the sides are of the same affection with the opposite angles that is, if the sides be greater or less than quadrants the opposite angles will be greater or less than right angles.

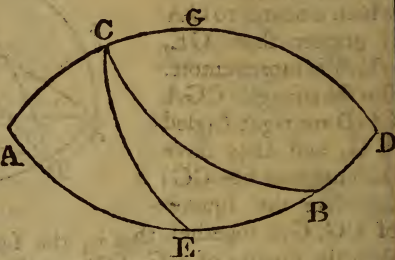
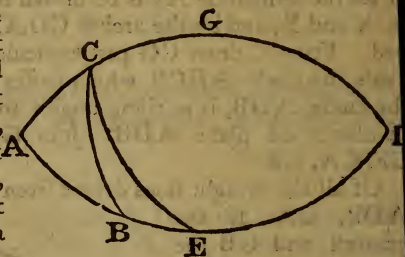
Let ABC be a spherical triangle, right angled at A , and side AB , will be of the same affection with the opposite angle ACB .

Case 1. Let AB be less than a quadrant. Let AE be a quadrant, and EC an arch of a great circle passing through E , C . Since A is a right angle, and AE a quadrant, E is the pole of the great circle AC , and ECA a right angle; but ECA is greater than BCA , therefore BCA is less than a right angle.

Q. E. D.

Case 2. Let AB be greater than a quadrant, make AE equal to a quadrant, and let a great circle pass through C , E . ECA is a right angle as before, and BCA is greater than ECA , that is, greater than a right angle.

Q. E. D.



P R O P.

P R O P. XV.

If the two sides of a right angled spherical triangle be of the same affection, the hypotenuse will be less than a quadrant; and if they be of different affection, the hypotenuse will be greater than a quadrant.

Let ABC be a right angled spherical triangle, if the two sides AB, AC be of the same or of different affection, the hypotenuse BC will be less or greater than a quadrant.

Case 1. Let AB, AC be each less than a quadrant. Let E, AG be quadrants; G will be the pole of AB, and E the pole of AC, and EC a quadrant; but, (Prop. 13.) CE greater than CB, since CB is farther off from CGD than E. In the same manner, it is shewn that CB, in the triangle CBD, where the two sides CD, BD are each greater than a quadrant, is less than CE, that is, less than a quadrant.

Case 2. Let AC be less, and AB greater than a quadrant; the hypotenuse BC will be greater than a quadrant; for, AE be a quadrant, then E is the pole of AC, and EC will be a quadrant. But CB is greater than CE (Prop. 13.), the AC passes through the pole of ABD. Q. E. D.

COR. 1. Hence, conversely, if the hypotenuse of a right angled triangle be greater or less than a quadrant, the sides will be of different or the same affection.

COR. 2. Since (Prop. 14.) the angles of a right angled spherical triangle have the same affection with the opposite sides, therefore, according as the hypotenuse is greater or less than a quadrant, the angles will be of different or of the same affection.

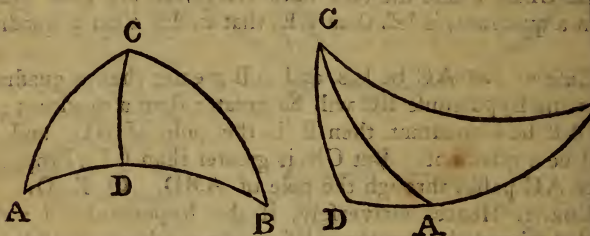
P R O P.

P R O P. XVI.

IN any spherical triangle, if the perpendicular on the base from the opposite angle fall within the triangle, the angles at the base are of the same affection; and if the perpendicular fall without the triangle, the angles at the base are of different affection.

Let ABC be a spherical triangle, and let the arch CD drawn from C perpendicular to the base AB .

1. Let CD fall within the triangle; then since ADC , BDC are right angled spherical triangles, the angles A , B must be of the same affection with CD (Prop. 14.).



2. Let CD fall without the triangle; then (Prop. 14.) the angle B is of the same affection with CD ; and the angle CAD is of the same affection with CD ; therefore angles CAD and B are of the same affection, and the angles CAB and B of different affections.

COR. Hence, if the angles A and B be of the same affection, the perpendicular will fall within the base; for, if it did not, A and B would be of different affection. And, if the angles A and B be of opposite affection, the perpendicular will fall without the triangle; for, if it did not, the angles A and B would be of the same affection, contrary to the supposition.

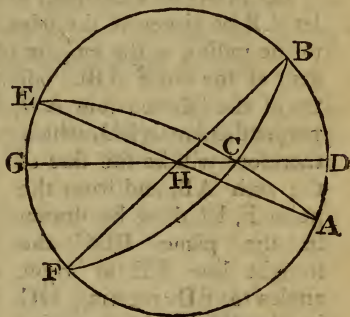
PROP. XVII.

to the base of a spherical triangle a perpendicular be drawn from the opposite angle, which either falls within the triangle, or is the nearest of two that fall without; the least of the segments the base is adjacent to the least of the sides of the triangle, or to the greatest, according as the angle of the sides is less or greater than a semicircle.

Let $ABEF$ be a great circle of a sphere, H its pole, and CD any circle passing through H , which therefore is perpendicular to the circle $ABEF$. Let A and B be two points on the circle $ABEF$ on opposite sides of the point D , and let E be nearer to A than to B , and let C be any point in the circle GND , between H and D . Through the points A and B and C , let the arches AC and BC be drawn, and let them be produced till they meet the circle $ABEF$ in the points E and F , then the arches ACE , BCF are semicircles. So ACB , ACF , CFE , ECB are four spherical triangles contained by arches of the same circles, and having the same perpendiculars CD and CG .

Now, because CE is nearer to the arch CHG than CB is, CE is greater than CB , and therefore CE and CA are greater than CB and CA , therefore CB and CA are less than a semicircle; but because CE is by supposition less than CB , AC is also less than CB , (prop. 13.), and therefore in this case, viz. when the perpendicular falls within the triangle, and when the sum of the sides is less than a semicircle, the least segment is adjacent to the least side.

Again, in the triangle FCA the two sides FC and CA are less than a semicircle; for, since AC is less than CB , AC and CF are less than BC and CF . Also, AC is less than CF , because it is more remote from CHG than CF is; therefore the least segment of the base AD is in this case also adjacent to the least side.



3. But

3. But in the triangle FCE the two sides FC and CE greater than a semicircle; for, since FC is greater than AC and CE are greater than AC and CE . And because AC is less than CB , EC is greater than CF , and EC is therefore nearer to the perpendicular CHG than CF is, where EG is the least segment of the base, and is adjacent to greater side.

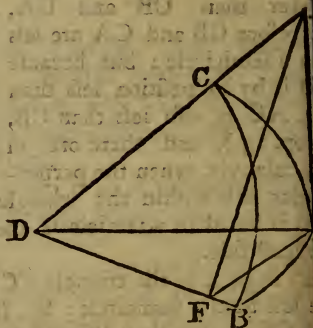
4. In the triangle ECB the two sides EC , CB are greater than a semicircle; for, since by supposition CB is greater than CA , EC and CB are greater than EC and CA . Also, EC is greater than CB , wherefore in this case, also, the least segment of the base EG is adjacent to the greatest side of triangle. Therefore, &c. **Q. E. D.**

P R O P. XVIII.

IN right angled spherical triangles, the sine of either of the sides about the right angle, is the radius of the sphere, as the tangent of the remaining side is to the tangent of the angle opposite to that side.

Let ABC be a triangle, having the right angle at A ; and let AB be either of the sides, the sine of the side AB will be the radius, as the tangent of the other side AC to the tangent of the angle ABC , opposite to AC . Let D be the centre of the sphere; join AD , BD , CD , and let AF be drawn perpendicular to BD , which therefore will be the sine of the arch AB , and from the point F , let there be drawn in the plane BDC the straight line FE at right angles to BD , meeting DC in E , and let AE be joined. Since therefore the straight line DF is at right angles to both FA and FE , it will also be at right angles to the plane AEF (4. 7.),

wherefore the plane ABD , which passes through D is perpendicular to the plane AEF (17. 7.), and the plane AEF perpendicular to ABD : But the plane ACD or AED is also perpendicular to the same ABD : Therefore the con-



non section, viz. the straight line AE , is at right angles to the plane ABD , (18. 7.), and EAF , EAD are right angles. Therefore AE is the tangent of the arch AC ; and in the rectilineal triangle AEF , having a right angle at A , AF is to the radius as AE to the tangent of the angle AFE , (1. Pl. Tr.); but AF is the sine of the arch AB , and AE the tangent of the arch AC , and the angle AFE is the inclination of the planes CBD , ABD , (4. def. 7.) or the spherical angle ABC : Therefore the sine of the arch AB is to the radius as the tangent of the arch AC to the tangent of the opposite angle ABC . Q. E. D.

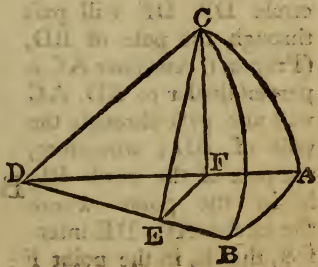
COR. And since by this proposition the sine of the side AB is to the radius, as the tangent of the other side AC to the tangent of the angle ABC opposite to that side; and as the radius is to the co-tangent of the angle ABC , so is the tangent of the same angle ABC to the radius, (2. Cor. def. 9. Pl. Tr.) by equality, the sine of the side AB is to the co-tangent of the angle ABC adjacent to it, as the tangent of the other side AC to the radius.

P R O P. XIX.

IN right angled spherical triangles the sine of the hypotenuse is to the radius, as the sine of either side is to the sine of the angle opposite to that side.

Let the triangle ABC be right angled at A , and let AC be either of the sides; the sine of the hypotenuse BC will be to the radius as the sine of the arch AC is to the sine of the angle ABC .

Let D be the centre of the sphere, and let CE be drawn perpendicular to DB , which will therefore be the sine of the hypotenuse BC ; and from the point E let there be drawn in the plane ABD the straight line EF perpendicular to DB and let CF be joined: CF will be at right angles to the plane ABD , as was shown in the preceding proposition of the straight line EA : Wherefore CFD , CFE are right angles, and CF is the sine of the arch AC ; and in the triangle CFE , having the right angle



angle CFE, CE is to the radius, as CF to the sine of the angle CEF, (1. Pl. Tr.). But, since CE, FE are at right angles DEB, which is the common section of the planes CBE, ABD, the angle CEF is equal to the inclination of the planes, (4. def. 7.), that is, to the spherical angle ABC. The sine, therefore, of the hypotenuse CB is to the radius as the sine of the side AC is to the sine of the opposite angle ABC. Q. E. D.

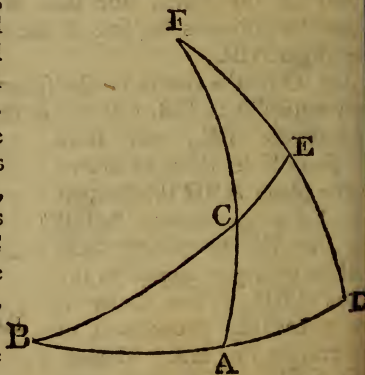
COR. Of these three, viz. the hypotenuse, a side, and the angle opposite to that side, any two being given, the third may be found.

P R O P. XX.

IN right angled spherical triangles, the co-sine of the hypotenuse is to the radius as the co-tangent of either of the angles is to the tangent of the remaining angle.

Let ABC be a spherical triangle, having a right angle at A, the co-sine of the hypotenuse BC will be to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB.

Describe the circle DE, of which B is the pole, and let it meet AC in F, and the circle BC in E; and since the circle BD passes through the pole B of the circle DF, DF will pass through the pole of BD, (Prop. 4.). And since AC is perpendicular to BD, AC will also pass through the pole of BD; wherefore, the pole of the circle BD is in the point where the circles AC, DE intersect, that is, in the point F: The arches FA, FD are there-



for

ore quadrants, and likewise the arches BD, BE: In the triangle CEF, right angled at the point E, CE is the complement of BC the hypotenuse of the triangle ABC, EF is the complement of the arch ED, which is the measure of the angle ABC, and FC the hypotenuse of the triangle CEF, is the complement of AC, and the arch AD, which is the measure of the angle CFE, is the complement of AB.

But (Prop. 18.) in the triangle CEF, the sine of the side CE is to the radius, as the tangent of the other side EF is to the tangent of the angle ECF opposite to it; that is, in the triangle ABC, the co-sine of the hypotenuse BC is to the radius as the co-tangent of the angle ABC is to the tangent of the angle ACB. Q. E. D.

COR. 1. Of these three, viz. the hypotenuse and the two angles, any two being given, the third will also be given.

COR. 2. And since by this proposition the co-sine of the hypotenuse BC is to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB, and since the radius is to the co-tangent of ACB, as the tangent of ACB to the radius, (Cor. 2. def. 9. Pl. Tr.); therefore *ex æquo*, the co-sine of the hypotenuse BC is to the co-tangent of the angle CB, as the co-tangent of the angle ABC to the radius.



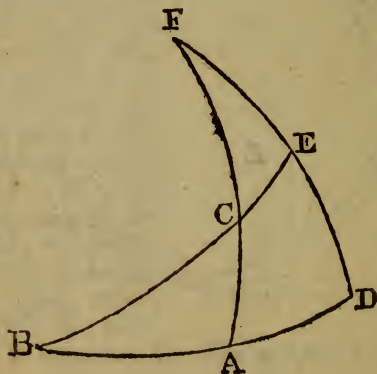
PROP.

P R O P. XXI.

IN right angled spherical triangles, the co-sine of an angle is to the radius, as the tangent of the side adjacent to that angle is to the tangent of the hypotenuse.

The same construction remaining: In the triangle CEF, (Prop. 18.) the sine of the side EF is to the radius, as the tangent of the other side CE is to the tangent of the angle CFE opposite to it, that is, in the triangle ABC, the co-sine of the angle ABC is to the radius as (the co-tangent of the hypotenuse BC to the co-tangent of the side AB, adjacent to ABC, or as) the tangent of the side AB to the tangent of the hypotenuse, since the tangents of two arches are reciprocally proportional to their co-tangents, (Cor. 2. def. 9 Pl. Tr.).

COR. And since by this proposition the co-sine of the angle ABC is to the radius, as the tangent of the side AB is to the tangent of the hypotenuse BC; and as the radius is to the co-tangent of BC, so is the tangent of BC to the radius; *ex æquo*, the co-sine of the angle ABC will be to the co-tangent of the hypotenuse BC, as the tangent of the side AB adjacent to the angle ABC to the radius.



P R O P.

P R O P. XXII.

In right angled spherical triangles, the co-sine of either of the sides is to the radius, as the co-sine of the hypotenuse is to the co-sine of the other side.

The same construction remaining: In the triangle CEF, the co-sine of the hypotenuse CF is to the radius, as the co-sine of the side CE to the co-sine of the opposite angle CFE, (Prop. XX); that is in the triangle ABC, the co-sine of the side CA is to the radius as the co-sine of the hypotenuse BC to the co-sine of the other side BA. Q. E. D.

P R O P. XXIII.

In right angled spherical triangles, the co-sine of either of the sides is to the radius, as the co-sine of the angle opposite to that side is to the co-sine of the other angle.

The same construction remaining: In the triangle CEF, the co-sine of the hypotenuse CF is to the radius as the co-sine of the side EF is to the co-sine of the angle ECF opposite to it, that is, in the triangle ABC, the co-sine of the side CA is to the radius, as the co-sine of the angle ABC opposite to it, is to the co-sine of the other angle ACB. Q. E. D.

SOLUTION of the CASES of Right Angled SPHERICAL TRIANGLES.

PROBLEM.

IN a right angled spherical triangle, of the three sides and three angles, any two being given, besides the right angle, to find the other three.

This problem has sixteen cases, the solutions of which are contained in the following table, where ABC is any spherical triangle right angled at A.

GIVEN.	SOUGHT.	SOLUTION.	
BC and B.	AC	$R : \text{fin. BC} :: \text{fin. B} : \text{fin. AC},$ (19).	1
	AB	$R : \text{cof. B} :: \tan. BC : \tan. AB,$ (21).	2
	C.	$R : \text{cof. BC} :: \tan. B : \text{co-t. C},$ (20).	3
AC and C.	AB.	$R : \text{fin. AC} :: \tan. C : \tan. AB,$ (18).	4
	BC.	$\text{cof. C} : R :: \tan. AC : \tan. BC,$ (21).	5
	B.	$R : \text{cof. AC} :: \text{fin. C} : \text{cof. B},$ (23).	6
AC and B.	AB.	$\tan. B : \tan. AC :: R : \text{fin. AB},$ (18).	7
	BC.	$\text{fin. B} : \text{fin. AC} :: R : \text{fin. BC},$ (19).	8
	C.	$\text{cof. AC} : \text{cof. B} :: R : \text{fin. C},$ (23).	9
AC and BC.	AB.	$\text{cof. AC} : \text{cof. BC} :: R : \text{cof. AB},$ (22).	10
	B.	$\text{fin. BC} : \text{fin. AC} :: R : \text{fin. B},$ (19).	11
	C.	$\tan. BC : \tan. AC :: R : \text{cof. C},$ (21).	12
AB and AC.	BC.	$R : \text{cof. AB} :: \text{cof. AC} : \text{cof. BC},$ (23).	13
	B.	$\text{fin. AB} : R :: \tan. AC : \tan. B,$ (18).	14
	C.	$\text{fin. AC} : R :: \tan. AB : \tan. C,$ (18).	14
B and C.	AB.	$\text{fin. B} : \text{cof. C} :: R : \text{cof. AB},$ (23).	15
	AC.	$\text{fin. C} : \text{cof. B} :: R : \text{cof. AC},$ (23).	15
	BC.	$\tan. B : \text{co-t. C} :: R : \text{cof. BC},$ (20).	16

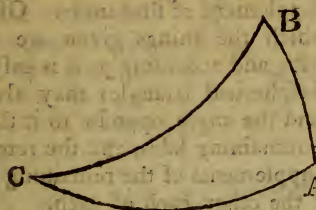


TABLE for determining when the things found in the preceding are less than a Quadrant.

The angle or arch found is less than 90° .

When B is less than 90° . 1

When BC and B are of the same affection. 2

When BC and B are of the same affection. 3

When C is less than 90° . 4

When AC and C are of the same affection. 5

When AC is less than 90° . 6

Ambiguous. 7

Ambiguous. 8

Ambiguous. 9

When AC and BC are of the same affection. 10

When AC is less than 90° . 11

When AC and BC are of the same affection. 12

When AB and AC are of the same affection. 13

When AC is less than 90° . 14

When AB is less than 90° . 14

When C is less than 90° . 15

When B is less than 90° . 15

When B and C are of the same affection. 16

The cases marked ambiguous are those in which the thing sought has two values, and may either be equal to a certain angle, or to the supplement of that angle. Of these there are three, in all of which the things given are a side, and the angle opposite to it; and accordingly, it is easy to shew, that two right angled spherical triangles may always be found that have a side and the angle opposite to it the same in both, but of which the remaining sides, and the remaining angle of the one, are the supplements of the remaining sides and the remaining angle of the other, each of each.

Though the affection of the arch or angle found may in all the other cases be determined by the rules in the second of the preceding tables, it may be useful to remark, that all the rules, except two, may be reduced to one, viz. that *when the thing found by the rules in the first table is either a tangent or a co-sine; and when, of the tangents or co-sines employed in the computation of it, one only belongs to an obtuse angle, the angle required is also obtuse.*

Thus, in the 15th case, when $\cos. AB$ is found, if C be an obtuse angle, because of $\cos. C$, AB must be obtuse; and in case 16. if either B or C be obtuse, BC is greater than 90° but if B and C are either both acute, or both obtuse, BC is less than 90° .

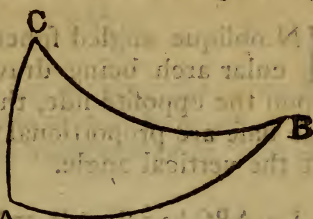
It is evident, that this rule does not apply when that which is found is the sine of an arch; and this, besides the three ambiguous cases, happens also in other two, viz. the 1st and 11th.

P R O P. XXIV.

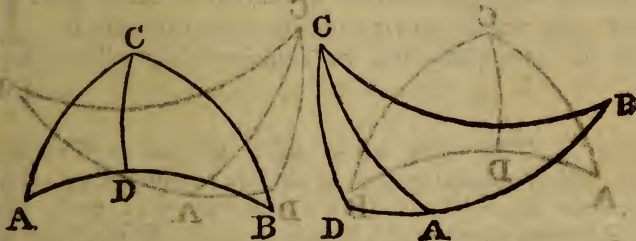
IN spherical triangles, whether right angled or oblique angled, the sines of the sides are proportional to the sines of the angles opposite to them.

First, Let ABC be a right angled triangle, having a right angle at A ; therefore, (Prop. 19.) the sine of the hypote-

the side BC is to the radius (or the fine of the right angle at A)
 as the fine of the side AC
 to the fine of the angle B.
 And, in like manner, the
 side of BC is to the fine of
 the angle A, as the fine of
 the side AB to the fine of the angle
 C; wherefore (II. 5.) the fine
 of the side AC is to the
 fine of the angle B, as the
 fine of AB to the fine of the angle C.



Secondly, Let ABC be an oblique angled triangle, the fine
 of any of the sides BC, will be to the fine of any of the
 other two AC, as the fine of the angle A opposite to BC, is
 to the fine of the angle B opposite to AC. Through the
 point C, let there be drawn an arch of a great circle CD per-
 pendicular upon AB; and in the right angled triangle BCD,
 (Prop. 19.) the fine of BC is to the radius, as the fine of



D to the fine of the angle B; and in the triangle ADC,
 by inversion, the radius is to the fine of AC as the fine
 of the angle A to the fine of DC: Therefore, *ex æquo pertur-*
batur, the fine of BC is to the fine of AC, as the fine of the
 angle A to the fine of the angle B. Q. E. D.

Z 4

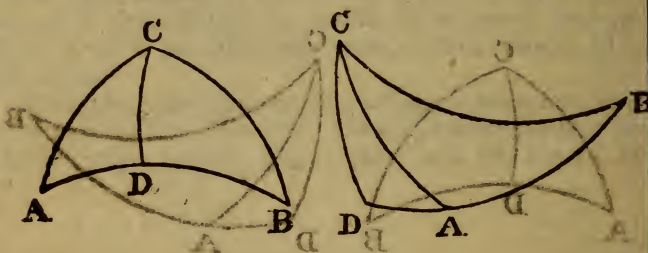
PROP.

P R O P. XXV.

IN oblique angled spherical triangles, a perpendicular arch being drawn from any of the angles upon the opposite side, the co-sines of the angles at the base are proportional to the sines of the segments of the vertical angle.

Let ABC be a triangle, and the arch CD perpendicular to the base BA ; the co-sine of the angle B will be to the co-sine of the angle A , as the sine of the angle BCD to the sine of the angle ACD .

For (Prop. 23.) the co-sine of the angle B is to the sine of the angle BC , as the co-sine of the side CD is to the radius and also the co-sine of the angle D to the sine of the angle ACD in the same ratio; therefore by permutation, the co-sine of the angle B is to the co-sine of the angle A , as the sine of the angle BCD to the sine of the angle ACD . Q. E. D.



P R O P. XXVI.

THE same things remaining, the co-sines of the sides BC , CA , are proportional to the co-sines of BD , DA , the segments of the base.

For (Prop. 22.) the co-sine of BC is to the co-sine of BD , as the co-sine of DC to the radius, and the co-sine of AC to the co-sine of AD in the same ratio: Wherefore, by permutation, the co-sines of the sides BC , CA are proportional to the co-sines of the segments of the base BD , DA . Q. E. D.

P R O P

P R O P. XXVII.

THE same construction remaining, the sines of BD, DA, the segments of the base are reciprocally proportional to the tangents of B and A, the angles at the base.

For (Prop. 18.) the sine of BD is to the radius, as the tangent of DC to the tangent of the angle B; and also, the radius to the sine of AD, as the tangent of A to the tangent of DC; Therefore, *ex æquo perturbatè*, the sine of BD is to the sine of DA, as the tangent of A to the tangent of B.

P R O P. XXVIII.

THE same construction remaining, the co-sines of the segments of the vertical angle are reciprocally proportional to the tangents of the sides.

For (Prop. 20.) the co-sine of the angle BCD, is to the radius, as the tangent of CD is to the tangent of BC; and also, (Prop. 21. by inversion), the radius is to the co-sine of the angle ACD, as the tangent of AC to the tangent of CD; Therefore, *ex æquo perturbatè*, the co-sine of the angle BCD is to the co-sine of the angle ACD, as the tangent of AC is to the tangent of BC. Q. E. D.

P R O P.

P R O P. XXIX.

IF from an angle of a spherical triangle there be drawn a perpendicular to the opposite side, or base, the rectangle contained by the tangents of half the sum, and of half the difference of the segments of the base is equal to the rectangle contained by the tangents of half the sum, and of half the difference of the two sides of the triangle.

Let ABC be a spherical triangle, and let the arch CD be drawn from the angle C at right angles to the base AB.

Because (Prop. 26.), $\text{cof. BC} : \text{cof. AC} :: \text{cof. BD} : \text{cof. AD}$, by composition and division,

$\text{cof. BC} + \text{cof. AC} : \text{cof. BC} - \text{cof. AC} :: \text{cof. BD} + \text{cof. AD} : \text{cof. BD} - \text{cof. AD}$. But, (1. Cor. 3. Pl. Trig.),

$\text{cof. BC} + \text{cof. AC} : \text{cof. BC} - \text{cof. AC} :: \cot. \frac{1}{2}(\text{BC} + \text{AC}) :$

$\tan. \frac{1}{2}(\text{BC} - \text{AC})$; and also, $\text{cof. BD} + \text{cof. AD} : \text{cof. BD} -$

$\text{cof. AD} :: \cot. \frac{1}{2}(\text{BD} + \text{AD}) : \tan. \frac{1}{2}(\text{BD} - \text{AD})$, therefore

$\cot. \frac{1}{2}(\text{BC} + \text{AC}) : \tan. \frac{1}{2}(\text{BC} - \text{AC}) :: \cot. \frac{1}{2}(\text{BD} + \text{AD}) :$

$\tan. \frac{1}{2}(\text{BD} - \text{AD})$. And because rectangles of the same alti-

tude are as their bases, therefore, $\tan. \frac{1}{2}(\text{BC} + \text{AC}) \times$

$\cot. \frac{1}{2}(\text{BC} + \text{AC}) : \tan. \frac{1}{2}(\text{BC} + \text{AC}) \times \tan. \frac{1}{2}(\text{BC} - \text{AC}) ::$

$\tan. \frac{1}{2}(\text{BD} + \text{AD}) \times \cot. \frac{1}{2}(\text{BD} + \text{AD}) : \tan. \frac{1}{2}(\text{BD} + \text{AD}) \times$

$\tan. \frac{1}{2}(\text{BD} - \text{AD})$. But $\tan. \frac{1}{2}(\text{BC} + \text{AC}) \times \cot. \frac{1}{2}(\text{BC} + \text{AC})$

$= R^2$, (2. Cor. def. 9. Pl. Tr) and also, $\tan. \frac{1}{2}(\text{BD} + \text{AD}) \times$

$\cot. \frac{1}{2}(\text{BD} + \text{AD}) = R^2$, therefore (9. 5.) $\tan. \frac{1}{2}(\text{BD} + \text{AD}) \times$

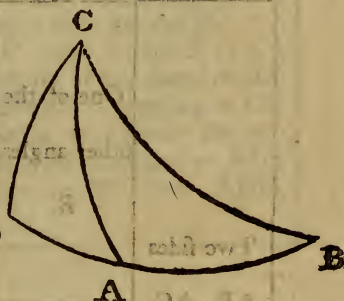
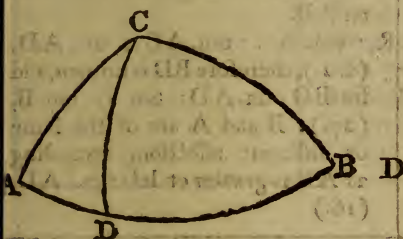
$\tan. \frac{1}{2}(\text{BD} - \text{AD}) = \tan. \frac{1}{2}(\text{BC} + \text{AC}) \times \tan. \frac{1}{2}(\text{BC} - \text{AC})$.

Q. E. D.

Cor.

COR. 1. Because the sides of equal rectangles are reciprocally proportional, $\tan. \frac{1}{2} (BD+AD) : \tan. \frac{1}{2} (BC+AC) :: \tan. \frac{1}{2} (BC-AC) : \tan. \frac{1}{2} (BD-AD)$.

COR. 2. Since, when the perpendicular CD falls within the triangle, $BD+AD=AB$, the base; and when CD falls without the triangle $BD-AD=AB$, therefore in the first case, the proportion in the last corollary becomes, $\tan. \frac{1}{2} (AB) : \tan. \frac{1}{2} (BC+AC) :: \tan. \frac{1}{2} (BC-AC) : \tan. \frac{1}{2} (BD-AD)$; and in the second case, it becomes by inversion and alternation, $\tan. \frac{1}{2} (AB) : \tan. \frac{1}{2} (BC+AC) :: \tan. \frac{1}{2} (BC-AC) : \tan. \frac{1}{2} (BD+AD)$.



SOLUTION

SOLUTION of the CASES of Oblique Angled SPHERICAL TRIANGLES.

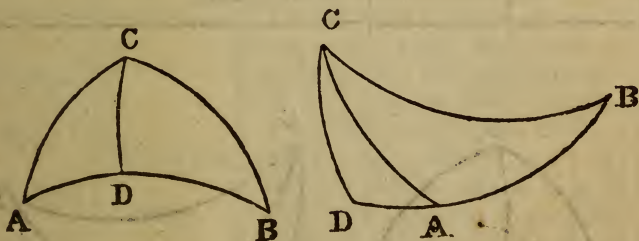
PROBLEM.

IN any oblique angled spherical triangle; of the three sides and three angles, any three being given, the other three may be found.

In this table, the references (c. 4.), (c. 5.), &c. are to the cases in the preceding tables.

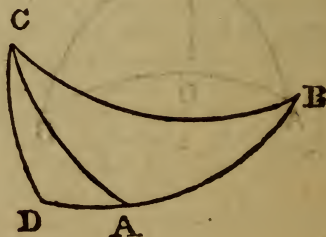
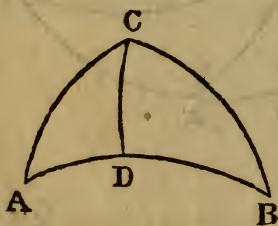
	GIVEN.	SOUGHT.	SOLUTION.
1	Two sides AB, AC, and the included angle A.	One of the other angles, B.	Let fall the perpendicular CD from the unknown angle not required, on AB. $R : \cos. A :: \tan. AC : \tan. AD,$ (c. 2.); therefore BD is known, and $\sin. BD : \sin. AD :: \tan. A : \tan. B,$ (27.); B and A are of the same or different affection, according as AB is greater or less than AD, (16.)
2		The third side BC.	Let fall the perpendicular CD from one of the unknown angles on the side AB. $R : \cos. A :: \tan. AC : \tan. AD,$ (c. 2.); therefore AD is known, and $\cos. AD : \cos. BD :: \cos. AC : \cos. BC,$ (26.); according as the segments AD and DB are of the same or different affection, AC and CB will be of the same or different affection.

GIVEN.	SOUGHT.	SOLUTION.
Two angles, A and ACB, and AC, the side be- tween them.	The side BC.	From C the extremity of AC next the side sought, let fall the perpendicular CD on AB. $R : \text{cof. AC} :: \tan. A : \text{cot. ACD}$, (c. 3.); therefore BCD is known, and $\text{cof. BCD} : \text{cof. ACD} :: \tan. AC : \tan. BC$, (28). BC is less or greater than 90° , according as the angles A and BCD are of the same or different affection.
	The third angle B.	Let fall the perpendicular CD from one of the given angles on the opposite side AB. $R : \text{cof. AC} :: \tan. A : \text{cot. ACD}$, (c. 3.); therefore the angle BCD is given, and $\text{fin. ACD} : \text{fin. BCD} :: \text{cof. A} : \text{cof. B}$, (25.); B and A are of the same or different affection, according as CD falls within or without the triangle, that is, according as ACB is greater or less than ACD, (16.)



GIVEN

	GIVEN.	SOUGHT.	SOLUTION.
5	Two sides AC and BC, and an angle A opposite to one of them, BC.	The angle B opposite to the other gi- ven side AC.	$\text{Sin. BC} : \text{fin. AC} :: \text{fin. A} : \text{fin. B}$, (24.) The affection of B is am- biguous, unless it can be deter- mined by this rule, that accord- ing as $\text{AC} + \text{BC}$ is greater or less than 180° , $A + B$ is greater or less than 180° , (10.).
6	AC and BC, and an angle A opposite to one of them, BC.	The angle ACB contained by the given sides AC and BC.	From ACB the angle sought draw CD perpendicular to AB; then $R : \text{cof. AC} :: \tan. A : \text{co-t. ACD}$, (c. 3.); and $\tan. BC : \tan. AC ::$ $\text{cof. ACD} : \text{cof. BCD}$, (28.). $\text{ACD} + \text{BCD} = \text{ACB}$, and ACB is ambiguous, because of the am- biguous sign + or -.
7	BC.	The third side AB.	Let fall the perpendicular CD from the angle C contained by the gi- ven sides upon the side AB. $R : \text{cof. A} :: \tan. AC : \tan. AD$, (c. 3.); $\text{cof. AC} : \text{cof. BC} ::$ $\text{cof. AD} : \text{cof. BD}$, (26.) $\text{AB} = \text{AD} + \text{BD}$, wherefore AB is ambiguous.



GIVEN.

GIVEN.	SOUGHT.	SOLUTION.
	The side BC opposite to the other given angle A.	$\text{Sin. } B : \text{fin. } A :: \text{fin. } AC : \text{fin. } BC,$ (24.); the affection of BC is uncertain, except when it can be determined by this rule, that according as $A+B$ is greater or less than 180° , $AC+BC$ is also greater or less than 180° , (10.).
Two angles A, B, and a side AC opposite to one of them,	The side AB adjacent to the given angles A, B.	From the unknown angle C draw CD perpendicular to AB; then $R : \text{cof. } A :: \text{tan. } AC : \text{tan. } AD,$ (c. 3.); $\text{tan. } B : \text{tan. } A :: \text{fin. } AD : \text{fin. } BD.$ BD is ambiguous, and therefore $AB = AD \pm BD$ may have four values, some of which will be excluded by this condition, that AB must be less than 180° .
B.	The third angle ACB.	From the angle required, C, draw CD perpendicular to AB. $R : \text{cof. } A :: \text{tan. } A : \text{co-t. } ACD,$ (c. 3.), $\text{cof. } A : \text{cof. } B :: \text{fin. } ACD : \text{fin. } BCD,$ (25.). The affection of BCD is uncertain, and therefore $ACB = ACD \pm BCD$ has four values, some of which may be excluded by the condition, that ACB is less than 180° .
The three sides, AB, AC; and BC.	One of the angles A.	From C one of the angles not required, draw CD perpendicular to AB. Find an arch E such that $\text{tan. } \frac{1}{2} AB : \text{tan. } \frac{1}{2} (AC+BC) :: \text{tan. } \frac{1}{2} (AC-BC) : \text{tan. } \frac{1}{2} E;$ then, if AB be greater than E, AB is the sum, and E the difference of AD and DB; but if AB be less than E, E is the sum, and AB the difference of AD, DB, (29.). In either case, AD and DB are known, and $\text{tan. } AC : \text{tan. } AD :: R : \text{cof. } A.$

	GIVEN.	SOUGHT.	SOLUTION.
12	The three angles A, B, C.	One of the sides BC.	Suppose the supplements of the three given angles, A, B, C, to be a, b, c , and to be the sides of a spherical triangle. Find, by the last case, the angle of this triangle opposite to the side a , and it will be the supplement of the side of the given triangle opposite to the angle A, that is, of BC, (10.); and therefore BC is found.

In the foregoing table the rules are given for ascertaining the affection of the arch or angle found, whenever it can be done: Most of these rules are contained in this one rule, which is of general application, viz. *that when the thing found is either a tangent or a co-sine, and of the tangents or co-sines employed in the computation of it, either one or three belong to obtuse angles, the angle found is also obtuse.* This rule is particularly to be attended to in cases 5. and 7. where it removes part of the ambiguity.

It may be necessary to remark with respect to the 11th case, that the segments of the base computed there are those cut off by the nearest perpendicular; and also, that when the sum of the sides is less than 180° , the least segment is adjacent to the least side of the triangle; otherwise to the greatest (Prop. 17.).

FINIS.

NOTES

ON THE

ELEMENTS.

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THE MUSEUM

NOTES

ON THE

FIRST BOOK OF THE ELEMENTS.

DEFINITIONS.

I.

IN the definitions a few changes have been made, of which Book I.
it is necessary that an account should be given. One of
these changes respects the first definition, that of a point,
which Euclid has said to be, 'That which has no parts, or
which has no magnitude.' Now, it has been objected to
this definition, that it contains only a negative, and that it is
not convertible, as every good definition ought certainly to be.
That it is not convertible is evident, for though every point
is unextended, or without magnitude, yet every thing unex-
tended, or without magnitude, is not a point. To this it is im-
possible to reply, and therefore it becomes necessary to change
the definition altogether, which is accordingly done here, a
point being defined to be, *that which has position but not mag-
nitude*. Here the affirmative part includes all that is essential

Book I. to a point, and the negative part excludes every thing that is not essential to it. I am indebted for this definition to a friend, by whose judicious and learned remarks I have often profited.

II.

After the second definition Euclid has introduced the following, 'the extremities of a line are points.'

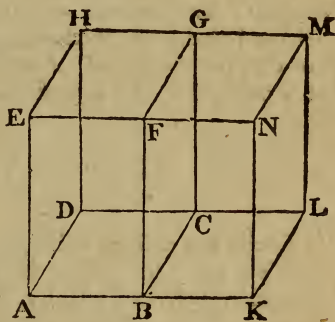
Now this is certainly not a definition, but an inference from the definitions of a point and of a line. For that which terminates a line can have no breadth, as the line in which it has none, and it can have no length, as it would not then be a termination, but a part of that which it is supposed to terminate. The termination of a line can therefore have no magnitude, and having necessarily position, it is a point. But as it is plain, that in all this we are drawing a consequence from two definitions already laid down, and not giving a new definition, I have taken the liberty of putting it down as a corollary to the second definition, and have added, that *the intersections of one line with another are points*, as this affords a good illustration of the nature of a point, and is an inference exactly of the same kind with the preceding. The same thing nearly has been done with the fourth definition, where that which Euclid gave as a separate definition, is made a corollary to the fourth, because it is in fact an inference deduced from comparing the definitions of a superficies and a line.

As it is impossible to explain the relation of a superficies, line and a point to one another, and to the solid in which they all originate, better than Dr Simson has done, I shall here add, with very little change, the illustration given by the excellent Geometer.

"It is necessary to consider a solid, that is, a magnitude which has length, breadth and thickness, in order to understand aright the definitions of a point, line and superficies; for these all arise from a solid, and exist in it: The boundary, or boundaries which contain a solid are called superficies, or the boundary which is common to two solids which are contiguous, or which divides one solid into two contiguous parts, is called a superficies: Thus, if BCGF be one of the boundaries which contain

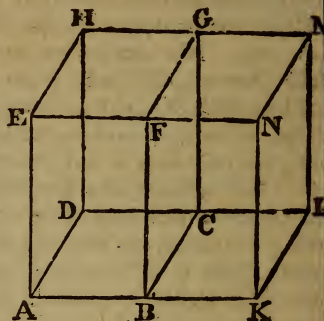
contain the solid $ABCDEFGH$, or which is the common boundary of this solid, and the solid $BKLCFNMG$, and is therefore in the one as well as the other solid, it is called a superficies, and has no thickness: For if it have any, this thickness must either be a part of the thickness of the solid AG , or the solid BM , or a part of the thickness of each of them. It cannot be a part of the thickness of the solid BM ; because, if this solid be removed from the solid AG , the superficies $BCGF$, the boundary of the solid AG , remains still the same as it was. Nor can it be a part of the thickness of the solid AG ; because, if this be removed from the solid BM , the superficies $BCGF$, the boundary of the solid BM , does nevertheless remain; therefore the superficies $BCGF$ has no thickness, but only length and breadth.

“The boundary of a superficies is called a line; or a line is the common boundary of two superficies that are contiguous, or is that which divides one superficies into two contiguous parts: Thus, if BC be one of the boundaries which contain the superficies $ABCD$, or which is the common boundary of this superficies, and of the superficies $KBCL$ which is contiguous to it, this boundary BC is called a line, and has no breadth: or, if it have any, this must be part either of the breadth of the superficies $ABCD$, or of the superficies $KBCL$, or part of each of them. It is not part of the breadth of the superficies $KBCL$; for, if this superficies be removed from the superficies $ABCD$, the line BC which is the boundary of the superficies $ABCD$ remains the same as it was. Nor can the breadth that BC is supposed to have, be a part of the breadth of the superficies $ABCD$; because, if this be removed from the superficies $KBCL$, the line BC , which is the boundary of the superficies $KBCL$, does nevertheless remain: Therefore the line BC has no breadth. And because the line BC is in a superficies, and that a superficies has no thickness, as was



{Book I. shewn; therefore a line has neither breadth nor thickness but only length.

“The boundary of a line is called a point, or a point is the common boundary or extremity of two lines that are contiguous: Thus, if B be the extremity of the line AB, or the common extremity of the two lines AB, KB, this extremity is called a point, and has no length: For, if it have any, the length must either be part of the length of the line AB, or of the line KB. It is not part of the length of KB; for, if the line KB be removed from AB, the point B which is the extremity of the line AB remains the same as it was: Nor is it part of the length of the line AB; for, if AB be removed from the line KB, the point B, which is the extremity of the line KB,



does nevertheless remain: Therefore the point B has no length: And because a point is in a line, and a line has neither breadth nor thickness, therefore a point has no length, breadth, nor thickness. And in this manner the definition of a point, line, and superficies are to be understood.”

III.

Euclid has defined a straight line to be a line which (as we translate it), “lies evenly between its extreme points.” This definition is obviously faulty, the word evenly standing as much in need of an explanation as the word straight, which it is intended to define. In the original, however, it must be confessed, that this inaccuracy is at least less striking than in our translation; for the word which we render *evenly* is *ἰσότητι*, *equally*, and is accordingly translated *ex æquo*, and *equaliter* by Gregory and Commandine. The definition, therefore, is that a straight line is one which lies equally between its extreme points; and if by this we understand a line that lies between

between its extreme points so as to be related exactly alike to the space on the one side of it, and to the space on the other, we have a definition that is perhaps a little too metaphysical, but which certainly contains in it the essential character of a straight line. That Euclid took the definition in this sense, however, is not certain, because he has not attempted to deduce from it any property whatsoever of a straight line, and indeed, it should seem not easy to do so, without employing some reasonings of a more metaphysical kind than he has any where admitted into his elements. To supply the defects of his definition, he has therefore introduced the Axiom, that *two straight lines cannot inclose a space*, on which Axiom it is, and not on his definition of a straight line, that his demonstrations are founded. As this manner of proceeding is certainly not so regular and scientific as that of laying down a definition, from which the properties of the thing defined may be logically deduced, I have substituted another definition of a straight line in the room of Euclid's. The definition of a straight line given here is taken from Boscovich, in his Notes on the Philosophical Poem of Professor Stay. In one of these he says, 'Rectam lineam rectæ congruere totam toti in infinitum productam si bina puncta unius binis alterius congruant, patet ex ipsa admodum clara rectitudinis idea quam habemus.' (Supplementum in lib. 3. § 550.) Now, that which Mr Boscovich would consider as an inference from our idea of straightness, seems itself to be the essence of that idea, and to afford the best criterion for judging whether any given line be straight or not. From this definition the Axiom above mentioned, viz. That two straight lines cannot inclose a space, follows as a necessary consequence. For, if two lines inclose a space, they must intersect one another in two points, and yet in the intermediate part must not coincide, and therefore by the definition they are not straight lines. It follows in the same way, that two straight lines cannot have a common segment, or cannot coincide in part, without coinciding altogether.

Since laying down the definition of a straight line, as in the text, I have been favoured by Dr Reid of Glasgow with the perusal of a MS. containing many excellent observations on the first Book of Euclid, such as might be expected from

Book I.

a philosopher distinguished for the accuracy as well as the extent of his knowledge. He there defines a straight line nearly as has been done here, viz. "A straight line is that which cannot meet another straight line in more points than one, otherwise they perfectly coincide, and are one and the same." Dr Reid also contends, that this must have been Euclid's own definition; because in the first proposition of the eleventh Book, that author argues, "that two straight lines cannot have a common segment, for this reason, that a straight line does not meet a straight line in more points than one, otherwise they coincide." Whether this amounts to a proof of the definition having been actually Euclid's, I will not take upon me to decide; but it certainly proves that it is a definition which the writings of that geometer ought long since to have suggested to his commentators; and it reminds me, that I might have learnt from these writings what I have acknowledged above as derived from a remoter source.

There is another characteristic, and obvious property of straight lines, by which I have often thought that they might be very conveniently defined, viz. that the position of the whole of a straight line is determined by the position of two of its points, in so much that, when two points of a straight line continue fixed, the line itself cannot change its position. It might therefore be said, that *a straight line is one in which, if the position of two points be determined, the position of the whole line is determined.* But this definition, though it amounts in fact just to the same thing with the former, is rather more abstract than it, and is not so easily made the foundation of reasoning; so that, after endeavouring as much as possible to accommodate it to the understanding of beginners, I have found it best to lay it aside, and to adopt the definition given in the text.

V.

The definition of a plane is given from Dr Simson, Euclid's being liable to the same objections with his definition of a straight line, for he says, that "a plane superficies is one which lies evenly between its extreme lines." The defects of this definition are completely removed in that which Dr Simson

son has given. Another definition different from both might have been adopted, viz. That those superficies are called plane, which are such, that if three points of the one coincide with three points of the other, the whole of the one must coincide with the whole of the other. This definition, as it resembles that of a straight line, already given, might, perhaps have been introduced with some advantage; but as the purposes of demonstration cannot be better answered than by that in the text, it has been thought best to make no farther alteration.

VI.

In Euclid, the general definition of a plane angle is placed before that of a rectilineal angle, and is meant to comprehend those angles which are formed by the meeting of other lines than straight lines. A plane angle is said to be 'the inclination of two lines to one another which meet together, but are not in the same direction.' This definition is omitted here, because that the angles formed by the meeting of curve lines, though they may become the subject of geometrical investigation, certainly do not belong to the Elements; for the angles that must first be considered are those made by the intersection of straight lines with one another. The angles formed by the contact or intersection of a straight line and a circle, or of two circles, or two curves of any kind with one another, could produce nothing but perplexity to beginners, and cannot possibly be understood till the properties of rectilineal angles have been fully explained. On this ground, without contesting the arguments which Proclus uses in defence of this definition, I have omitted it. Whatever is not useful, should, in explaining the elements of a science, be kept out of sight altogether; for, if it does not assist the progress of the understanding, it will certainly retard it.

A X I O M S.

Among the Axioms there have been made only two alterations. The 10th Axiom in Euclid is, that 'two straight lines cannot inclose a space;' which having become a corollary to our definition of a straight line, ceases of course to be ranked with self-evident propositions. It is therefore removed from among the Axioms, and that which was before the 11th is accounted the 10th.

The 12th Axiom of Euclid is, that 'If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced shall at length meet upon that side on which are the angles which are less than two right angles.' Instead of this proposition, which, though true, is by no means self-evident; another that appeared more obvious, and better entitled to be accounted an Axiom, has been introduced, viz. 'that two straight lines cannot be drawn through the same point parallel to the same straight line, without coinciding with one another.' On this subject, however, a fuller explanation is necessary, for which see the note on the 29th Prop.

P R O P. IV. and VIII. B. I.

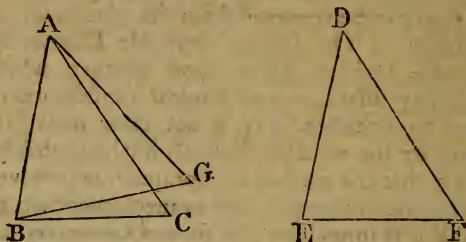
The fourth and eighth propositions of the first book are the foundation of all that follows with respect to the comparison
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of triangles. They are demonstrated by what is called the method of supraposition, that is, by laying the one triangle upon the other, and proving that they must coincide. To this some objections have been made, as if it were ungeometrical to suppose one figure to be removed from its place and applied to another figure. 'The laying,' says Mr Thomas Simpson in his Elements, 'of one figure upon another, whatever evidence it may afford, is a *mechanical* consideration, and depends on no postulate.' It is not clear what Mr Simpson meant here by the word *mechanical*; but he probably intended only to say, that the method of supraposition involves the idea of motion, which belongs rather to mechanics than geometry; for, I think it is impossible that such a Geometer as he was could mean to assert, that the evidence derived from this method is like that which arises from the use of instruments, and of the same kind with what is furnished by experience and observation. The demonstrations of the fourth and eighth, as they are given by Euclid, are as certainly a process of pure reasoning, depending solely on the idea of equality, as established in the 8th Axiom, as any thing in geometry. But, if still the removal of the triangle from its place be considered as creating a difficulty, and as inelegant, because it involves an idea, that of motion, not essential to geometry, this defect may be entirely remedied, provided that, to Euclid's three postulates, we be allowed to add the following, viz. *That if there be two equal straight lines, and if any figure whatsoever be constituted on the one, a figure every way equal to it may be constituted on the other.* Thus, if AB and DE be two equal straight lines, and ABC a triangle on the base AB, a triangle DEF every way equal to ABC may be supposed to be constituted on DE as a base. By this it is not meant to assert that the method of describing the triangle DEF is actually known, but merely that the triangle DEF may be conceived to exist in all respects equal to the triangle ABC. Now, there is no truth whatsoever that is better entitled than this to be ranked among the Postulates or Axioms of geometry; for the straight lines AB and DE being every way equal, there can be nothing belonging to the one that may not also belong to the other.

On the strength of this postulate the fourth Prop. is thus demonstrated.

If

Book I. If ABC, DEF be two triangles, such that the two sides AB and AC of the one are equal to the two ED, DF of the



other, and the angle BAC , contained by the sides AB, AC of the one, equal to the angle EDF , contained by the sides ED, DF of the other; the triangles ABC and EDF are every way equal.

On AB let a triangle be constituted every way equal to the triangle DEF ; then, if this triangle coincide with the triangle ABC , it is evident that the proposition is true, for it is equal to DEF by hypothesis, and to ABC , because it coincides with it; wherefore ABC, DEF are equal to one another. But if it does not coincide with ABC , let it have the position ABG ; and first, let G not fall on AC ; then the angle BAG is not equal to the angle BAC . But the angle BAG is equal to the angle EDF , therefore EDF and ABC are not equal, and they are also equal by hypothesis, which is impossible. Therefore the point G must fall upon AC ; now, if it fall upon AC but not at C , then AG is not equal to AC ; but AG is equal to DF , therefore DF and AC are not equal, and they are also equal by supposition, which is impossible. Therefore G must coincide with C , and the triangle AGB with the triangle ACB . But AGB is every way equal to DEF , therefore ACB and DEF are also every way equal. Q. E. D.

In the same manner may the 8th be demonstrated, and even with more facility, so that these two fundamental Theorems may be proved without having recourse to the method of supposition. Such demonstrations, it must, however, be acknowledged, trespass against a rule which Euclid has uniformly adhered to throughout the Elements, except where
he

was forced by necessity to depart from it. This rule is, that nothing is ever supposed to be done, the manner of doing which has not been already taught, and the construction derived either directly from the three postulates laid down in the beginning, or from some problems already reduced to those postulates. Now, this rule is not essential to geometrical demonstration, where, for the purpose of discovering the properties of figures, we are certainly at liberty to suppose any figure to be constructed, or any line to be drawn, the existence of which does not involve an impossibility. The use, therefore, of Euclid's rule is to guard against the introduction of impossible hypotheses, or the taking for granted that a thing may exist which in fact implies a contradiction; from such suppositions, false conclusions might, no doubt, be deduced, and this rule is therefore useful as far as it answers the purpose of excluding them. But the foregoing postulatam could never lead to suppose the actual existence of any thing that is impossible; for it only supposes the existence of a figure equal and similar to one already existing, but in a different part of space from it, or, to speak more precisely, having one of its sides in an assigned position. Unless therefore there be an impossibility in the existence of the one of these figures, there can be none in that of the other. This new postulate might, therefore, it would seem, be introduced with good effect into the Elements of Geometry; but to have adopted it here would have led us too far from the text of Euclid, and it is sufficient for the present purpose to have pointed it out.

P R O P. VII.

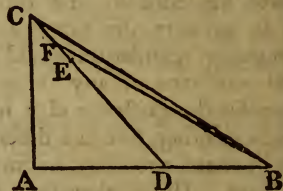
Dr Simson has very properly changed the enunciation of this proposition, which, as it stands in the original, is considerably embarrassed and obscure. His enunciation, with very little variation, is retained here.

P R O P. XXI.

It is essential to the truth of this proposition, that the straight lines drawn to the point within the triangle be drawn from the two extremities of the base; for, if they be drawn from

Book I. from other points of the base, their sum may exceed the sum of the sides of the triangle in any ratio that is less than that of two to one. This is demonstrated by Pappus Alexandrinus in the 3d Book of his *Mathematical Collections*, but the demonstration is of a kind that does not belong to this place. If it be required simply to shew, that in certain cases the sum of the two lines drawn to the point within the triangle may exceed the sum of the sides of the triangle, the demonstration is easy, and is given nearly as follows by Pappus, and also by Proclus, in the 4th Book of his Commentary on Euclid.

Let ABC be a triangle, having the angle at A a right angle; let D be any point in AB; join CD, then CD will be greater than AC, because in the triangle ACD, the angle CAD is greater than the angle ADC. From DC cut off DE equal to AC; bisect AE in F, and join BF; BF and FD are greater than BC and CA.



Because CF is equal to FE, CF and FB are equal to EF and FB, but CF and FB are greater than BC, therefore EF and FB are greater than BC. To EF and FB add ED, and to BC add AC, which is equal to ED by construction, and BF and FD will be greater than BC and CA. Q. E. D.

It is evident, that if the angle BAC be obtuse, the same reasoning may be applied.

This proposition is a sufficient vindication of Euclid for having demonstrated the 21st proposition, which some affect to consider as self-evident; for it proves, that the circumstance on which the truth of that proposition depends is not obvious, nor that which at first sight it is supposed to be, viz. that of the one triangle being included within the other. For this reason I cannot agree with M. Clairault, that Euclid demonstrated this proposition only to avoid the cavils of the Sophists. But I must, at the same time, observe, that what the former has said on the subject has certainly been misunderstood; and in one respect, unjustly censured by Dr Simson. The exact translation of his words is as follows: "If Euclid has taken the trouble to demonstrate, that a triangle included within another has

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the sum of its sides less than the sum of the sides of the triangle in which it is included, we are not to be surpris'd. That geometer had to do with those obstinate Sophists, who made a point of refusing their assent to the most evident truths," &c. (Elemens de Geometrie par M. Clairault. Pref.)

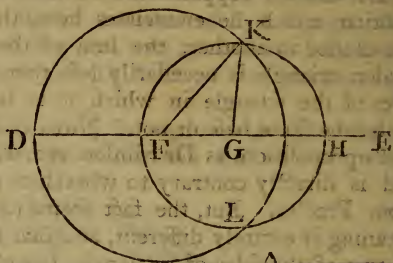
Dr Simson supposes M. Clairault to mean by the proposition which he enunciates here, that when one triangle is included in another, the sum of the two sides of the included triangle is necessarily less than the sum of the two sides of the triangle in which it is included, whether they be on the same base or not. Now, this is not only not Euclid's proposition, as Dr Simson remarks, but it is not true, and is directly contrary to what has just been demonstrated from Proclus. But, the fact seems to be, that M. Clairault's meaning is entirely different, and that he intends to speak not of two of the sides of a triangle, but of all the three, so that his proposition is, "that when one triangle is included within another, the sum of all the three sides of the included triangle is less than the sum of all the three sides of the other," and this is without doubt true, though, I think, by no means self-evident. It must be acknowledged also, that it is not exactly Euclid's proposition, which, however, it comprehends under it, and is the general theorem, of which the other is only a particular case. Therefore, though M. Clairault may be blamed for maintaining that to be an Axiom which requires demonstration, yet he ought not to be accused of mistaking in his instance a false proposition for a true one.

P R O P. XXII.

Thomas Simpson in his Elements has objected to Euclid's demonstration of this proposition, because it contains no proof, that the two circles made use of in the construction of the Problem must cut one another; and Dr Simson, on the other hand, always unwilling to acknowledge the smallest blemish in the works of Euclid, contends, that the demonstration is perfect. The truth, however, certainly is, that the demonstration admits of some improvement; for the limitation that is

Book I. is made in the enunciation of any Problem ought always to be shewn to be necessarily connected with the construction of it, and this is what Euclid has neglected to do in the present instance. The defect may easily be supplied, and Dr Simson himself has done it in effect in his note on this proposition though he denies it to be necessary.

Because that of the three straight lines DF, FG, GH, any two are greater than the third, by hypothesis, FD is less than FG and GH, that is, than FH, and therefore the circle described from the centre F with the distance FD must meet the line FE between F and H; and for the like reason, the circle



described from the centre G at the distance GH, must meet DG between D and G, and therefore, the one of these circles cannot be wholly within the other. Neither can the one be wholly without the other, because DF and GH are greater than FG; the two circles must therefore intersect one another.

P R O P. XXIV.

The subject of parallel lines is one of the most difficult in the Elements of Geometry. It has accordingly been treated of in a great variety of different ways, of which, perhaps there is none that can be said to have given entire satisfaction. The difficulty consists in *converting* the 27th and 28th of Euclid, or in demonstrating, that parallel straight lines, or such as do not meet one another, when they meet a third line, make the alternate angles with it equal, or which comes to the

the same, are equally inclined to it, and make the exterior angle equal to the interior and opposite. In order to demonstrate this proposition, Euclid assumed it as an Axiom, that if a straight line meet two straight lines, so as to make the interior angles on the same side of it less than two right angles, these straight lines being continually produced, will at length meet on the side on which the angles are that are less than two right angles.' This proposition, however, is not self-evident, and ought the less to be received without proof, that, as Proclus has observed, the converse of it is a proposition that confessedly requires to be demonstrated. For the converse of it is, that two straight lines which meet one another make the interior angles, with any third line, less than two right angles; or, in other words, that the two interior angles of any triangle are less than two right angles, which is the 17th of the First Book of the Elements; and it should seem, that a proposition can never rightly be taken for an Axiom, of which the converse requires a demonstration.

This defect in Euclid is, therefore, abundantly evident, but the manner of correcting it is by no means obvious, as will appear from a review of the methods chiefly employed for that purpose.

These have been of three sorts: 1. A new definition of parallel lines: 2. A new manner of reasoning on the properties of straight lines without any new Axiom: 3. The introduction of a new Axiom, less exceptionable than Euclid's.

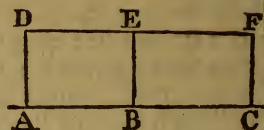
1. One of the definitions that has been substituted for Euclid's is, that straight lines are parallel, which preserve always the same distance from one another, by the word distance being understood, a perpendicular drawn to one of the lines from any point whatever in the other. If these perpendiculars be every where of the same length, the straight lines are called parallel. This is the definition given by Wolfius, by Boscovich, and by Thomas Simpson, in the first edition of his Elements. It is, however, a very faulty definition, for it conceals an Axiom in it, and takes for granted a property of straight lines, that ought either to be laid down as self-evident, or demonstrated, if possible, as a Theorem. Thus, if

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Book I.

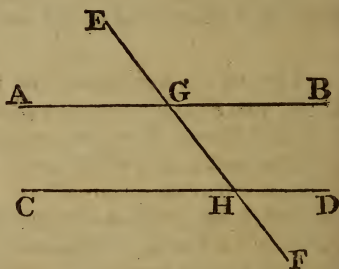
from the three points A, B, and C of the straight line AC, perpendiculars AD, BE, CF be drawn all equal to one another, it is implied in the definition, that the points D, E and F are in the same straight line, which, though it be true, it was not the business of the definition to inform us of. Two



perpendiculars, as AD and CF, are alone sufficient to determine the position of the straight line DF, and therefore the definition ought to be, "that two straight lines are parallel, when there are two points in the one, from which the perpendiculars drawn to the other are equal, and on the same side of it."

This is the definition of parallels which M. D'Alembert seem to prefer to all others; but he acknowledges, and very justly, that it still remains a matter of difficulty to demonstrate, that all the perpendiculars drawn from the one of these lines to the other are equal. (*Encyclopedie, Art. Parallele.*)

Another definition that has been given of parallels is, that they are lines which make equal angles with a third line, toward the same parts, or such as make the exterior angle equal to the interior and opposite. Varignon, Bezout, and several other mathematicians have adopted this definition, which, it must be acknowledged, is a perfectly good one, if it be understood by it, that the two lines, called parallel, are such as make equal angles with a certain third line, but not with any line that falls upon them. For, if this last is supposed, we, in fact, involve a Theorem in the definition, viz. That if AB and CD make equal angles with GH, they will



do so also with any other line whatsoever, which is a proposition to be proved. But, the definition, when thus explained, has no advantage whatever above Euclid's; and it still remains to demonstrate, 1. That straight lines which are equally inclined to a certain line that falls upon them, must be equally inclined to all the others that fall upon them; and al-

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fo, 2. That straight lines which do not meet when produced, must make equal angles with every line that falls upon them. On the whole, therefore, we may conclude, that of the three definitions that may be given of parallels, Euclid's, and the other two just mentioned, no one, as to the object of facilitating the demonstration of the properties of such lines, has any advantage above the rest.

Next, with respect to those who have sought to demonstrate the properties of parallel lines by help of any of the preceding definitions, without any new Axiom, all their attempts, as far as I know, have been unsuccessful. Of these I shall, however, mention only two, the one the most ancient, and the other the most recent of which I have any information.

The first is by Ptolemy the astronomer, who, as Proclus informs us, wrote an entire Book in proof of the proposition which Euclid has considered as an Axiom, concerning the meeting of such lines, as make with another line the interior angles less than two right angles. Proclus has preserved an account of this work, in the fourth book of his Commentaries, from which it appears, that Ptolemy had endeavoured to give his demonstrations without introducing any new Axiom; and it is remarkable, that he reasons there exactly on the principle which the moderns have distinguished by the name of the *sufficient reason*. To prove, that if two parallel straight lines, AB and CD, be cut by a third line EF, in G and H, the two interior angles AGH, CHG will be equal to two right angles, he reasons thus: If the angles AGH, CHG be not equal to two right angles, let them, if possible, be greater than two right angles; then, because the lines AG and CH are not more parallel than the lines BG and DH, the angles BGH, DHG are also greater than two right angles. Therefore, the four angles AGH, CHG, BGH, DHG are greater than four right angles; and they are also equal to four right angles, which is absurd. In the same manner it is shewn, that the angles AGH, CHG cannot be less than two right angles. Therefore they are equal to two right angles.

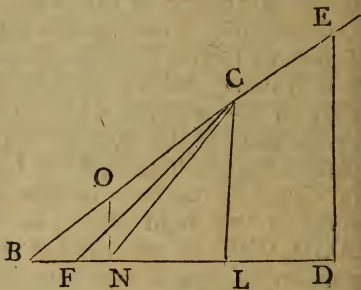
Proclus objects to this reasoning, and it certainly cannot be admitted as conclusive. For why are we to suppose that the interior angles which the parallels make with the line cutting them, are either in every case greater than two right

Book I.

angles, or in every case less than two right angles? For any thing that we are yet supposed to know, they may be sometimes greater than two right angles, and sometimes less, and therefore we are not entitled to conclude, because the angles AGH, CHG are greater than two right angles, that therefore the angles BGH, DHG are also necessarily greater than two right angles. It may safely be asserted, therefore, that Ptolemy has not succeeded in his attempt to demonstrate the properties of parallel lines without the assistance of a new Axiom.

The other attempt to demonstrate the same proposition without the assistance of a new Axiom is, by a modern geometer, Franceschinis, Professor of Mathematics in the University of Bologna, in an essay, which he entitles, *La Teoria delle parallele rigorosamente dimostrata*, printed in his *Opuscoli Matematici*, at Bassano in 1787.

The difficulty is there reduced to a proposition nearly the same with this, That if BE make an acute angle with BD, and if DE be perpendicular to BD at any point, BE and DE, if produced, will meet. To demonstrate this, it is supposed, that BO, BC are two parts taken in BE, of which BC is greater than BO, and that the perpendiculars ON, CL are drawn to BD; then shall BL be greater than BN. For, if not, that is,



if the perpendicular CL falls either at N, or between B and N, as at F; in the first of these cases the angle CNB is equal to the angle ONB, because they are both right angles, which is impossible; and, in the second, the two angles CFN, CNF, of the triangle CNF, exceed two right angles. Therefore, adds our author, since, as BC increases, BL also increases, and since BC may be increased without limit, so BL may become greater than any given line, and therefore may be greater than BD; wherefore, since the perpendiculars to BD from points beyond D meet BC, the perpendicular from D necessarily meets it. Q. E. D.

Now, though at first sight this reasoning may appear sound. it will be proved, that whatever multiple BC is of BO, the unless BL of BN, the indefinite increase of BC does not necessarily imply the indefinite increase of BL, or that BL may exceed BD. On the contrary, BL may always be less, and yet may do so in such a manner as never to exceed BD: In order that the demonstration should be conclusive, it would be necessary to shew, that when BC increases by a part equal to BO, BL increases always by a part equal to BN; but to do this will be found to require the knowledge of those very properties of parallel lines that we are seeking to demonstrate.

The foregoing proposition, therefore, is not demonstrated; if it were, the doctrine of parallel lines would be subject to no farther difficulty, as will appear from the demonstration of Clavius.

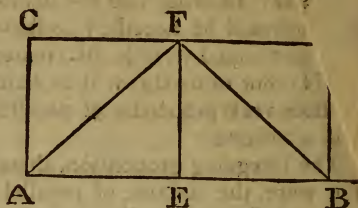
Clavius is one of the geometers who has treated most fully of this subject, but it may be doubted, whether he considered his demonstration as being founded on a new Axiom, or not. The proposition that serves as the foundation of his reasoning he certainly does not lay down entirely as self evident, for he offers a sort of metaphysical proof of it, by which he endeavours to connect it with Euclid's definition of a straight line. In this he is not successful, and, by what he says at the conclusion of the whole, he appears to have been willing to wave this proof, and to consider the said proposition as an Axiom; for he says, "*Quamvis autem, concesso principio nostro optime a nobis demonstratum sit tertium decimum hoc Axioma, (sc. Euclidis,)*" &c.; where he certainly speaks of the principle of his demonstration as a thing taken for granted. It is this, "That a line of which the points are all equidistant from a certain straight line in the same plane with it, is itself a straight line." From this proposition, which, however, ought not in strictness to be held an Axiom, it follows, that if two equal perpendiculars be drawn to any straight line on the same side of it, the straight line joining their extremities will be equidistant from the former line. For, if not, it is plain from it, that two straight lines may comprehend a space.

Book I.

Hence, if through the extremities C and D ^{of two per-}pendiculars, AC and BD, to the same straight ^{ne AB, a}line CD be drawn, CD shall make right ^{es with}angles with the perpendiculars AC and BD.

For, if AB be bisected in E, and EF be drawn ^{per-}pendicular to AB meeting CD in F; and if AF, BF be ^{pendi-}drawn,

because the sides AE, EF of the triangle AEF are equal to the sides BE, EF of the triangle BEF, and because they contain the equal angles AEF, BEF, each of which is a right angle, therefore the base AF is equal to the base BF,



and the angle EFA to the angle EFB, as also the angle EAF to the angle EBF. Now, since the whole angle CAE is equal to the whole DBE, and the part EAF to the part EBF, therefore the remainder FAC is equal to the remainder FBD. Therefore, since AC is equal to BD, and AF to BF; and, since the angles CAF, DBF are also equal, the triangles ACF, BDF are equal, the angle ACF to the angle BDF, and the angle AFC to the angle BFD. Now, it has been shewn, that the angle AFE is equal to the angle BFE, therefore the whole angle CFE is equal to the whole angle DFE, and therefore CFE, DFE are each of them right angles. But, because AC and EF are equal perpendiculars drawn to the line AE, and CF passes through their extremities, the angles ACF, CFE will be shewn to be equal, in the same way that the angles ACF and FDB have just been shewn to be equal. But CFE is a right angle, therefore ACF is also a right-angle. For the same reason BDC is a right angle.

Hence, lastly, the demonstration of Euclid's 12th Axiom, viz. that when a straight line, falling on two other straight lines, makes the internal angles, on the same side of it, less than two right angles, the two lines will meet, if continually towards that side on which the two angles are the less than two right angles.

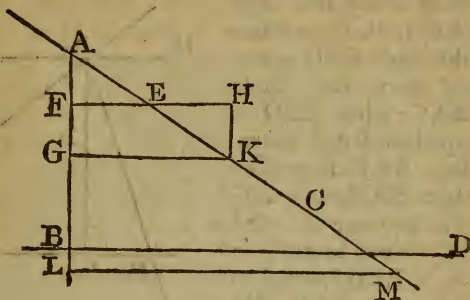
Let AC and BD be two straight lines, on which AB falls and makes the angles CAB, ABD less than two right angles, AC and BD will meet, if produced on the side of C and D.

First, let ABD be a right angle, and therefore CAB an acute angle. In AC take any point E,

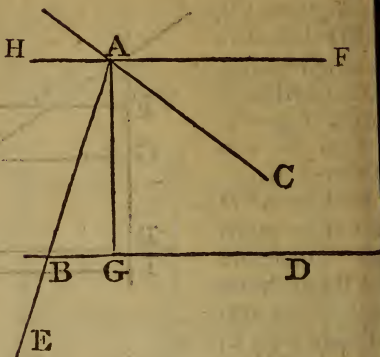
and draw EF perpendicular to AB; also, let AL be a multiple of AF such as to exceed AB, and let AM be the same multiple of AE that AL is of AF. Take FG equal to AF, and EK equal to EA, produce FE to H, so that EH may be equal to EF, and join HK; join also LM.

Because AE is equal to EK, FE to EH, and the angle AEF to the angle KEH, the triangles AFE, KHE are equal, the base HK to the base AF, and the angle EHK to the angle AFE. But AFE is a right angle, wherefore EHK is also a right angle, and HK being equal to AF, that is, to FG, KGF is a right angle, by the last proposition, and KG a perpendicular to AB; and, as the same may be proved of all the lines joining the corresponding points of division in AB and AC, it is true also of LM, and therefore LM is perpendicular to LA. But BD is also perpendicular to LA, therefore LM and BD are parallel, and the whole of the line LM is therefore on the same side of BD, namely, on the side of it opposite to A, therefore the line MA, that is, AC, must cut the line BD.

Next, Let the angles BAC and ABD, which the two lines make with the third line, be neither of them right angles. Through A draw HAF making the angle BAF equal to the angle EBD, and make AG perpendicular to BD; and, because the angles BAC, ABD are less than two right angles, and the angles EBD, ABD equal to two right angles, the



Book I. angles EBD and ABD are together greater than the angle CAB, ABD, and taking away the angle ABD, there remains the angle EBD greater than the angle BAC; but EBD is equal to BAF, therefore BAF is greater than BAC, and AC falls between AF and BD. Now, because the angle EBD is equal to the angle BAF, BD is parallel to AF; and because AG is at



right angles to BD, it is also at right angles to AF; for, not, the angle GAF will either be acute or obtuse, but it is not acute, for then, by the last case, the straight lines AC and BD would meet toward C and D, which is absurd, for AF and BD are parallel. Neither is the angle GAF obtuse, for the angle HAG would then be acute, and the straight lines AH and GB would meet, if produced toward B and H. Since the angle GAF, therefore, is a right angle, the angle GAC is less than a right angle, and therefore, by the last case, AC and GD will meet, if produced continually toward C and D. Q. E. D.

It is thus that Clavius demonstrates Euclid's Axiom, and of consequence, the whole doctrine of parallel lines, in a manner altogether unexceptionable, if the principle be taken for granted, "that a line, of which the points are all equidistant from a certain straight line, in the same plane with it, is itself a straight line." A demonstration, not very different from the preceding, is given by Dr Simson in the second, and the subsequent editions of his Euclid. In his first edition he followed a method which he admits not to be strictly demonstrative, grounded on this, that when two straight lines cut one another, they may be produced till their distance become greater than any given line, a principle which Proclus has also employed for the same purpose. In his other demonstration

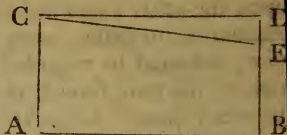
demonstration he differs from Clavius, first, as to the principle which he takes for granted; which is, "that a straight line cannot first come nearer to another straight line, and then go farther from it without meeting it;" where, by *coming nearer*, is understood, according to a previous definition, the diminution of the perpendiculars drawn from the one line upon the other; and the contrary by *going farther off*. This Axiom is, no doubt, more readily assented to than that of Clavius, and must be regarded as the more fundamental proposition of the two, though there be, in fact, no great difference between them. It is evident how it may be employed to demonstrate Clavius's first Theorem, viz. That if the perpendiculars drawn upon one straight line from two points in another be equal, all the perpendiculars drawn to the first of these lines from the points of the second are equal to one another. For otherwise the perpendiculars must first increase and then decrease, which, by the Axiom, is impossible. After this the demonstration proceeds in the same manner with that which has just been given, except that in some parts the proof is rendered direct where Clavius has it indirect, as in the second case of the last proposition. On the whole, however, the two demonstrations are so nearly alike, that as the one of them has been fully explained, the other does not require a particular consideration.

These, then, are some of the most remarkable demonstrations of the properties of parallel lines, that have been given, so far as I know, by ancient or modern geometers. I shall mention only one more, which I cannot help regarding as superior in simplicity and neatness to any of them. It is that which is given by Mr Thomas Simpson, in the second edition of his Elements, which is very different from that of the first edition, where he lays down the faulty definition of parallel lines that was already taken notice of. The Axiom employed by Simpson is essentially the same with that which was last mentioned, but it is presented in a simpler form. It is, that "if two points in a straight line are posited at unequal distances from another straight line in the same plane, those two lines being indefinitely produced on the side of the least distance will meet one another."

By help of this Axiom it is easy to prove, that if two straight lines AB, CD be parallel, the perpendiculars to the one, terminated by the other, are all equal, and are also perpendicular to

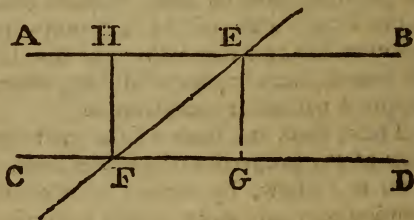
Book I. to both the parallels. That they are equal is evident, otherwise the lines would meet by the Axiom: That they are perpendicular to both is demonstrated thus:

If AC and BD, which are perpendicular to AB, and equal to one another, be not also perpendicular to CD, from C let CE be drawn at right angles to BD. Then, because AB and CE are both perpendicular to BD, they are parallel, and therefore, the perpendiculars AC and BE are equal. But AC is equal to BD, (by hypothesis), therefore BE and BD are equal, which is impossible; BD is therefore at right angles to CD.



Hence the proposition, that "if a straight line fall on two parallel lines, it makes the alternate angles equal," is easily derived. Let FH

and GE be perpendicular to CD, then they will be parallel to one another, and also at right angles to AB, and therefore FG and HE are equal to one another, by the last



proposition. Wherefore in the triangles EFG, EFH, the sides HE and EF are equal to the sides GF and FE, each to each, and also the third side HF to the third side EG, therefore the angle HEF is equal to the angle EFG, and they are alternate angles. Q. E. D.

This method of treating the doctrine of parallel lines is extremely plain and concise, and perhaps does not admit of any improvement, except that of putting the Axiom on which it is founded into a more simple form: It might, for example, be expressed thus: "If two straight lines intersect one another, and if from any two points of the one, perpendiculars be drawn to the other; the perpendicular that is near-

er to the point of intersection is less than that which is more remote."

The demonstration founded on this Axiom, and conducted the manner just explained, I am disposed to think preferable to any that has yet been given. If I have not followed this method, it is because I wished to preserve the text of Euclid with the least alteration possible. I therefore assumed as Axiom, a proposition that is not perhaps so obvious a property of straight lines as that which has just been stated, but which is certainly much more so than Euclid's, and one which I know from experience that beginners find no difficulty in comprehending, or in admitting to be true. By means of it the 29th, and also what was formerly the 12th Axiom, are demonstrated without changing any thing in the series of Euclid's propositions.

From the detail that has just been given, it is evident, that to demonstrate the properties of parallel lines without having recourse to some Axiom, or which is the same thing, without assuming some property of straight lines, not contained in the definition of a straight line, is still a desideratum in elementary geometry. And, if we consider how much skill and industry have in the course of many ages been applied to this investigation; and also reflect, that the thing sought for belongs to the very rudiments of the science, and therefore, if it exists at all, can be at no great distance, we shall be inclined to consider the discovery of it, as a problem in geometry never likely to be resolved. At the same time, it appears extraordinary, that the definition of a straight line, if it is complete, should not lead us to the knowledge of every property of such a line, without the assumption of any thing not contained in it. Why ought not the proposition, for instance, that has been just stated as an Axiom, that "if two straight lines intersect one another, and if from any two points in the one perpendiculars be drawn to the other, the perpendicular nearer to the point of intersection will be less than that which is more remote," to be capable of demonstration, or of being deduced from the definition of a straight line? If there be nothing obscure or imperfect in our notions of a straight line, of a perpendicular, or of the intersection of two straight lines, from whence can it possibly arise that we are unable to demonstrate this proposition? It was no way wonderful, that, when Euclid gave his vague and obscure definition

Book 1.

nition of a straight line, he should not be able to demonstrate even the most simple property of rectilineal figures without the assumption of the Axiom, that two straight lines cannot inclose a space. But, when the defects of the definition seem to be wholly corrected, we might certainly expect to be able to derive all the properties of straight lines directly from that definition, and yet the fact is, that we are not at all assisted by it, in the case before us. I confess this is not easily accounted for; but there are two considerations which, though they may not contain the solution of the paradox, will perhaps serve to render it less wonderful.

The first is, that the definition given of an angle is certainly imperfect. An angle is the *inclination* of two lines; now, the word inclination is not much better understood than the word angle, so that we have here the very same defect that there is in Euclid's definition of a straight line. It is, at the same time, difficult to conceive any way in which this definition can be amended; and it is, perhaps, on account of its imperfection that we are obliged to assume some property of the line subtending an angle, or of two lines making angles with a third line, as an Axiom; just as the imperfect definition of a straight line must be assisted by the assumption, that two straight lines cannot inclose a space.

It must farther be remarked, that whatever be the source of this difficulty, it is not the only one of the kind that we meet with in the Elements of Geometry. A second instance occurs, where a certain relation between the lengths of straight, and curve lines is assumed as an Axiom, without being logically deduced from our ideas, either of straightness, or of curvature. This is the Axiom on which Archimedes, and all the geometers after him have founded the comparison of the lengths of curves with the lengths of straight lines, and is the same which is placed here at the beginning of the 8th Book. It would be in vain, I believe, that one would seek to give a rigorous demonstration of that proposition, yet it is of a nature purely elementary, though more complex without doubt, than that which we have been considering. There is a third proposition of this sort, relative to surfaces, which is also laid down by Archimedes, for the foundation of the comparison of curve superficies with plane figures, and of which no demonstration is given. These are the only three properties

properties of geometrical magnitudes in the whole science, Book I.
 t are taken for granted without being deduced from the
 initions; that it is impossible to demonstrate any of them,
 what no one will take upon him to affirm; but the many
 l powerful efforts made for that purpose, which they have
 eady withstood, ought to deter any one from throwing a-
 y much of his time in searching after such demonstra-
 us.

BOOK II. PROP. VII.

[THE demonstration of this proposition usually occasions Book II.
 some difficulty to beginners, and on that account ano-
 er is added, which is somewhat shorter.

PROP. A and B.

These Theorems are added on account of their great use in
 ometry, and their close connection with the other proposi-
 ons which are the subject of this Book. Prop. A is an ex-
 nsion of the 9th and 10th.

BOOK III.

Book III.

B O O K I I I .

D E F I N I T I O N S .

THE definition which Euclid makes the first of this Book is that of equal circles, which he defines to be “the of which the diameters are equal.” This is rejected from among the definitions, as being a Theorem, the truth of which is proved by supposing the circles applied to one another, so that their centres may coincide, for the whole of the one must then coincide with the whole of the other. The converse, viz. That circles which are equal have equal diameters, is proved in the same way.

The definition of the angle of a segment is also omitted because it is not a rectilineal angle, but one understood to be contained between a straight line and a portion of the circumference of a circle. For the same reason, no notice is taken in the 16th proposition of the angle comprehended between the semicircle and diameter, which is said by Euclid to be greater than any acute rectilineal angle. The reason for these omissions has already been assigned in the notes on the fifth definition of the first Book.

B O O K V .

Book V.

THE subject of proportion has been treated so differently by those who have written on elementary geometry, and the method which Euclid has followed has been so often and so inconsiderately censured, that in these notes it will not perhaps be more necessary to account for the changes that have made, than for those that I have not made. The changes are but few, and relate to the language, not to the essence of the demonstrations

emonstrations; they will be explained after some of the definitions have been particularly considered. Book V.

DEF. III.

The definition of ratio given here has been greatly extolled by some authors; but whatever value it may have in the eyes of a metaphysician, it has but little in those of a geometer, because nothing concerning the properties of ratios can be deduced from it. Dr Barrow has very judiciously remarked concerning it, "That Euclid had probably no other design in making this definition, than to give a general summary idea of ratio to beginners by premising this metaphysical definition, to the more accurate definitions of ratios that are equal to one another, or one of which is greater or less than the other: I call it a metaphysical, for it is not properly a mathematical definition, since nothing in mathematics depends on it, or is deduced, nor, as I judge, can be deduced from it." (Barrow's Lectures, lect. 3.) Dr Simson thinks the definition has been added by some unskilful editor, but there is no ground for that supposition, other than what arises from the definition being of no use; we may, however, well enough imagine, that a certain idea of order and method induced Euclid to give some general definition of ratio, before he used the term in the definition of equal ratios.

DEF. IV.

This definition is a little altered in the expression; Euclid has it, that "magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the greater."

DEF. V.

One of the chief obstacles to the ready understanding of the 5th Book of Euclid is the difficulty that most people find of

Book V. of reconciling the idea of proportion which they have already acquired, with the account of it that is given in the definition. Our first ideas of proportion, or of proportionality, are got by trying to compare together the magnitude of external bodies; and, though they be at first abundantly vague and incorrect, they are usually rendered tolerably precise by the study of arithmetic; and we then learn to distinguish a particular relation, in point of magnitude, which being in any case the same between two things that it is between other two, we say of all the four, that they are proportionals.

The simplest form in which this relation can appear is, when the first contains the second the same number of times that the third contains the fourth; or, as we may express it, when the first is the same multiple of the second that the third is of the fourth. From this similitude of relations between the first two and the second two, we call them proportionals, or say, that the first is to the second as the third to the fourth.

Proceeding to generalise this idea, and taking number in the first place for the subject of comparison, we soon come to fix on this as the general notion of proportional numbers, that when there are four numbers, such that the quotient arising from dividing, according to the common rules of arithmetic the first by the second, is the same with the quotient that arises from dividing, in like manner, the third by the fourth, these numbers are proportionals, or, we say that the first is to the second as the third to the fourth.

Now, as the operation of arithmetical division applies as readily to any two magnitudes, of the same kind, as it does to two numbers, the notion of proportion thus obtained may be considered as perfectly general. For, in arithmetic, after finding how often the divisor is contained in the dividend, we multiply the remainder by 10, or 100, or 1000, or any power, as it is called, of 10, and proceed to enquire how oft the divisor is contained in this new dividend; and, if there be any remainder, we go on to multiply it by 10, 100, &c. as before, and to divide the product by the original divisor, and so on, the division sometimes terminating, by no remainder being left, and sometimes going on *ad infinitum*, in consequence of a remainder being left at each operation. Now, this process may easily be imitated with any two magnitudes A and B, providing

providing they be of the same kind, or such that the one can be multiplied so as to exceed the other. For, suppose that B is the least of the two; take B out of A as oft as it can be found, and let the quotient be noted, and also the remainder, if there be any; multiply this remainder by 10, or 100, &c. so as to exceed B, and let B be taken as oft as it can be found out of the quantity produced by this multiplication, and let the quotient be noted, and also the remainder, if there be any. Proceed with this remainder as before, and so on continually; and it is evident, that we have an operation that applies to all magnitudes whatsoever, and that may be performed with respect to any two lines, any two plane figures, or any two solids, &c.

Now, when we have two magnitudes and two others, and find that the first divided by the second, according to this method, gives the very same series of quotients that the third does when divided by the fourth, we say of these magnitudes, as we did of the numbers above described, that the first is to the second as the third to the fourth. There are only two more circumstances to be taken notice of as necessary to give to this idea of proportion, the utmost generality of which it is capable.

First, it is known from arithmetic, that the multiplication of the successive remainders each of them by 10, is equivalent to multiplying the quantity to be divided by the product of all those tens; so that multiplying, for instance, the first remainder by 10, the second by 10, and the third by 10, is the same thing, with respect to the quotient, as if the quantity to be divided had been at first multiplied by 1000; and therefore, our standard of the proportionality of numbers may be expressed thus: If the first multiplied any number of times by 10, and then divided by the second, gives the same quotient as when the third is multiplied as often by 10 and then divided by the fourth, the four magnitudes are proportionals.

Again, it is evident, that there is no necessity in these multiplications for confining ourselves to 10, or the powers of 10, and that we do so, in arithmetic, only for the conveniency of the decimal notation; we may therefore use any numbers whatsoever. providing we use the same in both cases. Hence, we have this definition of proportionals, when there are four magnitudes, and any multiple whatsoever of the first, when divided by the second, gives the same quotient

Book V.

with the like multiple of the third when divided by the fourth, the four magnitudes are proportionals, or the first has the same ratio to the second that the third has to the fourth.

We are now arrived very near to Euclid's definition; for let A, B, C, D be four proportionals, according to the definition just given, and m any number; and let the multiple of A by m , that is mA , be divided by B ; and, first, let the quotient be the number n exactly, then also, when mC divided by D , the quotient will be n exactly. But, when mA divided by B gives n for the quotient, $mA = nB$ by the nature of division, so that when $mA = nB$, $mC = nD$, which is one of the conditions of Euclid's definition.

Again, when mA is divided by B , let the division not be exactly performed, but let n be the quotient as far as it can be expressed in whole numbers, then $mA > nB$; and, for the same reason, $mC > nD$, which is another of the conditions of Euclid's definition.

Lastly, Let us suppose, that mA does not contain B so often as there are units in n , then mA will be less than nB ; but because, by hypothesis, mC does not contain D oftener than mA contains B , mC will not contain D so often as there are units in n , and therefore $mC < nD$. Therefore, we call A, B, C, D proportionals, when they are such, that if $mA > nB$, $mC > nD$; if $mA = nB$, $mC = nD$; and if $mA < nB$, $mC < nD$, m and n being any numbers whatsoever. Now, this is exactly the criterion of proportionality established by Euclid in the 5th definition, and is derived hereby by generalising the common and most familiar idea of proportion.

It appears from this, that the condition of mA containing B , whether with or without a remainder, as often as mC contains D , with or without a remainder, and of this being the case whatever value be assigned to the number m , includes in it all the three conditions that are mentioned in Euclid's definition; and hence, that definition may be expressed a little more simply by saying, that four magnitudes are proportionals, when any multiple of the first contains the second (with or without remainder), as oft as the same multiple of the third contains the fourth. But, though this definition is certainly, in the expression, more simple than Euclid's, it is not, as will be found on trial, so easily applied to the purpose of demonstration. The three conditions which Euclid brings together

together in his definition, though they somewhat embarrass the expression of it, have the advantage of rendering the demonstrations more simple than they would otherwise be, by avoiding all discussion about the magnitude of the remainder left, after B is taken out of mA as oft as it can be found. All the attempts, indeed, to demonstrate the properties of proportionals rigorously, by means of other definitions than Euclid's, either that I have made myself, or that I have seen others make, have only served to convince me of the excellence of the method followed by the Greek geometer, and of his singular success in generalising the idea of proportion.

The great objection to the other methods is, that if they are meant to be rigorous, they require two demonstrations to every proposition, one when the division of mA into parts equal to B can be exactly performed, the other when it cannot be exactly performed, whatever value be assigned to m , or when A and B are what is called incommensurable; and this last case will in general be found to require an indirect demonstration, or a *reductio ad absurdum*.

M. D'Alembert, speaking of the doctrine of proportion, in a discourse that contains many excellent observations, but in which he has overlooked Euclid's manner of treating this subject entirely, has the following remark: "On ne peut démontrer que de cette manière, (la réduction à l'absurde) la plupart des propositions qui regardent les incommensurables. L'idée de l'infini entre au moins implicitement dans la notion de ces sortes de quantités; et comme nous n'avons qu'une idée négative de l'infini on ne peut démontrer directement, *et a priori*, tout ce qui concerne l'infini mathématique." (*Encyclopedie. Mot, Geometrie.*)

This remark sets in a strong and just light the difficulty of demonstrating the propositions that regard the proportion of incommensurable magnitudes, without having recourse to the *reductio ad absurdum*, but it is surprising that M. D'Alembert, a geometer no less learned than profound, should have neglected to make mention of Euclid's method, the only one in which the difficulty he states so strongly is completely overcome. It is overcome by the introduction of the idea of infinitude, (if I may be permitted to use the word), instead of the idea of infinity; for m and n , the multipliers employed, are supposed to be indefinite, or to admit of all possible values, and it is by the skilful use of this condition that the ne-

Book V. cessity of indirect demonstrations is avoided. In the whole of geometry, I know not that any happier invention is to be found; and it is worth remarking, that Euclid appears in another of his works to have availed himself of the idea of indefinitude with the same success, viz. in his books of Porisms, which, have been restored by Dr Simson, and in which the whole analysis turned on that idea, as I have shewn at length, in the third volume of the Transactions of the Royal Society of Edinburgh. The investigations of those propositions were founded entirely on the principle of certain magnitudes admitting of innumerable values; and the methods of reasoning concerning them seem to have been extremely similar to those employed in the fifth of the Elements. It is curious to remark this analogy between the different works of the same author; and to consider, that the skill, in the conduct of this very refined and ingenious method, which Euclid had acquired in treating the properties of proportionals, may have enabled him to succeed so well in treating the still more difficult subject of Porisms.

With such an opinion of Euclid's manner of treating proportion, as I have now expressed, it was impossible that I should attempt to change any thing in the principle of his demonstrations. I have only sought to improve the language of them, by introducing a concise mode of expression, of the same nature with that which we use in arithmetic, and in algebra. Ordinary language conveys the ideas of the different operations supposed to be performed in these demonstrations so slowly, and breaks them down into so many parts that they make not a sufficient impression on the understanding. This, indeed, will generally happen when the things treated of are not represented to the senses by Diagrams, as they cannot be when we reason concerning magnitude in general as in this part of the Elements. Here we ought certainly to adopt the language of arithmetic or algebra which, by its shortness, and the rapidity with which it places objects before us, makes up in the best manner possible for being merely a conventional language, and using symbols that have no resemblance to the things expressed by them. Such a language, therefore, I have endeavoured to introduce here; and, I am convinced, that if it shall be found an improvement, it is the only one of which the fifth of Euclid

Euclid will admit. In other respects I have followed Dr Simson's edition, to the accuracy of which it would be difficult to make any addition.

In one thing I must observe, that the doctrine of proportion, as laid down here, is meant to be more general than in Euclid's Elements. It is intended to include the properties of proportional numbers as well as of all magnitudes. Euclid has not this design, for he has given a definition of proportional numbers in the seventh Book, very different from that of proportional magnitudes in the fifth; and it is not easy to justify the logic of this manner of proceeding; for we can never speak of two numbers and two magnitudes both having the same ratios, unless the word ratio have in both cases the same signification. All the propositions about proportionals here given are therefore understood to be applicable to numbers; and accordingly, in the eighth Book, the proposition that proves equiangular parallelograms to be in a ratio compounded of the ratios of the numbers proportional to their sides, is demonstrated by help of the propositions of the fifth Book.

On account of this, the word quantity, rather than magnitude, ought in strictness to have been used in the enunciation of these propositions, because we employ the word quantity to denote, not only things extended, to which alone we give the name of magnitudes, but also numbers. It will be sufficient, however, to remark, that all the propositions respecting the ratios of magnitudes relate equally to all things of which multiples can be taken, that is, to all that is usually expressed by the word quantity in its most extended signification, taking care always to observe, that ratio takes place only among like quantities. (See Def. 4.)

D E F. X.

The definition of compound ratio was first given accurately by Dr Simson; for, though Euclid used the term, he did so without defining it. I have placed this definition before those of duplicate and triplicate ratio, as it is in fact more general, and as the relation of all the three definitions is best seen

Book V. when they are ranged in this order, and expressed in the manner done here; duplicate ratio being called a ratio compounded of two equal ratios, and triplicate of three equal ratios, &c.

It was justly observed by Dr Simson, that the expression *compound ratio* is introduced merely to prevent circumlocution and for the sake principally of enunciating those propositions with conciseness that are demonstrated by reasoning *ex æquo* that is, by reasoning from the 22d or 23d of this Book. This will be evident to any one who considers carefully the Prop. 16 of this, or the 23d of the 6th Book.

B O O K VI.

DEFINITION II.

Book VI. **T**HIS definition is changed from that of *reciprocal figures*, which was of no use, to one that corresponds to the language used in the 14th and 15th propositions, and in other parts of geometry.

P R O P. XXVII. XXVIII. XXIX.

As considerable liberty has been taken with these propositions, it is necessary that the reasons for doing so should be explained. In the first place, when the enunciations are translated literally from the Greek, they sound very harshly, and are, in fact, extremely obscure. The phrase of applying to a straight line, a parallelogram deficient, or exceeding by another parallelogram, is so elliptical and so little analogous to ordinary language, that there could be no doubt of the propriety of at least changing the enunciations.

It next occurred, that the problems themselves in the 28th and 29th propositions are proposed in a more general form than is necessary in an elementary work, and that, therefore, to take those cases of them that are the most useful, as they happen to be the most simple, must be the best way of accommodating them to the capacity of a learner. The problem which Euclid proposes in the 28th is, "To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram;" which also might be more intelligibly enunciated thus: "To cut a given line, so that the parallelogram that has in it a given angle, and that is contained under one of the segments of the given line, and a straight line which has a given ratio to the other segment, may be equal to a given space;" instead of which problem I have substituted this other; "to divide a given straight line so that the rectangle under its segments may be equal to a given space." In the actual solution of problems, the greater generality of the former proposition is an advantage more apparent than real, and is fully compensated by the simplicity of the latter, to which, also, it is always easily reducible.

The same may be said of the 29th, which Euclid enunciates thus: "To a given straight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram similar to a given parallelogram." This might be proposed otherwise; "to produce a given line, so that the parallelogram having in it a given angle, and contained by the whole line produced, and a straight line that has a given ratio to the part produced, may be equal to a given rectilineal figure." Instead of this, is given the following problem, more simple, and, as was observed in the former instance, very little less general: "To produce a given straight line, so that the rectangle contained by the segments, between the extremities of the given line, and the point to which it is produced, may be equal to a given space."

Book VI.

P R O P. A, B, C, &c.

There are eight propositions added to this Book, on account of their utility and their connection with this part of the elements. The first four of them are in Dr Simson's edition, and among these Prop. A is given immediately after the third, being, in fact, a second case of that proposition, and capable of being included with it, in one enunciation. Prop. D is remarkable for being a theorem of Ptolemy the astronomer, in his *Μεγάλη Συνταξις*, and the foundation of the construction of his trigonometrical tables. Prop. E is the simplest case of the former; it is also useful in trigonometry, and, under another form, was the 97th, or, in some editions, the 94th of Euclid's Data. The propositions F and G are very useful properties of the circle, and are taken from the *Loci Plani* of Apollonius. Prop. H is a very remarkable property of the triangle.

B O O K VII.

Book VII. The reason for departing from Euclid in the geometry of solids has been already explained in the Preface, so that it only remains to make a few remarks on some particular definitions and theorems.

D E F. VIII. and P R O P. XX.

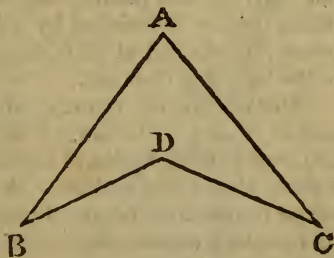
Solid angles, which are defined here in the same manner as in Euclid, are magnitudes of a very peculiar kind, and may be remarked for not admitting of that accurate comparison, one with another, which is common in the other subjects of geometry.

geometry. It cannot, for example, be said of one solid angle, that it is the half, or the double of another solid angle, nor did any geometer ever think of proposing the problem of bisecting a given solid angle. In a word, no multiple or submultiple of such an angle can be taken, and we have no way of expounding, even in the simplest cases, the ratio which one of them bears to another.

In this respect, therefore, a solid angle differs from every other magnitude that is the subject of mathematical reasoning, all of which have this common property, that multiples and submultiples of them may be found. It is not our business here to enquire into the reason of this anomaly, but it is plain, that on account of it, our knowledge of the nature and the properties of such angles can never be very far extended, and that our reasonings concerning them, must be chiefly confined to the relations of the plane angles, by which they are contained. One of the most remarkable of those relations is that which is demonstrated in the 20th of this Book, and which is, that all the plane angles which contain any solid angle must together be less than four right angles. This proposition is the 21st of the 11th of Euclid.

This proposition, however, is subject to a restriction in certain cases, which, I believe, was first observed by M. le Sage of Geneva, in a communication to the Academy of Sciences of Paris in 1756. When the section of the pyramid formed by the planes that contain the solid angle, is a figure that has none of its angles exterior, such as a triangle, a parallelogram, &c. the truth of the proposition just enunciated cannot be questioned. But, when the aforesaid section is a figure like that which is annexed, viz.

ABDC, having some of its angles, such as BDC, exterior, or, as they are sometimes called, re-entering angles, the proposition is not necessarily true; and it is plain, that in such cases the demonstration which we have given, and which



is the same with Euclid's, will no longer apply. Indeed, it were easy to shew, that on bases of this kind, by multiplying the number of sides, solid angles

Book VII.

angles may be formed, such that the plane angles which contain them shall exceed four right angles by any quantity assigned. An illustration of this from the properties of the sphere is perhaps the simplest of all others. Suppose that, on the surface of a hemisphere there is described a figure, bounded by any number of arches of great circles making angles with one another, on opposite sides alternately, the plane angles at the centre of the sphere that stand on these arches may evidently exceed four right angles, and that, too, by multiplying and extending the arches, in any assigned ratio. Now, these plane angles contain a solid angle at the centre of the sphere, according to the definition of a solid angle.

We are to understand the proposition in the text, therefore, to be true only of those solid angles in which the inclination of the plane angles are all the same way, or all directed toward the interior of the figure. To distinguish this class of solid angles from that to which the proposition does not apply, it is perhaps best to make use of this criterion, that they are such, that when any two points whatsoever are taken in the planes that contain the solid angle, the straight line joining those points falls wholly within the solid angle; or thus, they are such, that a straight line cannot meet the planes which contain them in more than two points. It is thus, too, that I would distinguish a plane figure that has none of its angles exterior, by saying, that it is a rectilineal figure, such that a straight line cannot meet the boundary of it in more than two points.

We distinguish, therefore, solid angles into two species, one in which the bounding planes can be intersected by a straight line only in two points; and another where the bounding planes may be intersected by a straight line in more than two points; to the first of these the proposition in the text applies, to the second it does not.

Whether Euclid meant entirely to exclude the consideration of figures of the latter kind, in all that he has said of solids, and of solid angles, it is not now easy to determine: It is certain, that his definitions involve no such exclusion; and as the introduction of any limitation would considerably embarrass these definitions, and render them difficult to be understood by a beginner, I have left it out, reserving to this place the full explanation of the difficulty. I cannot conclude this note without remarking, with the historian of the

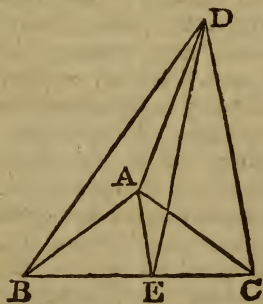
Academy,

Academy, that it is extremely singular, that not one of all those who had read or explained Euclid before M. le Sage, appears to have been sensible of this mistake. (*Memoires de l'Acad. des Sciences* 1756, *Hist.* p. 77.) A circumstance that renders this still more singular is, that another mistake of Euclid on the same subject, and perhaps of all other geometers, escaped M. le Sage also, and was first discovered by Dr Simson, as will appear in the following note.

D E F. IX. and P R O P. XXI.

These relate to similar and equal solids, a subject on which mistakes have prevailed not unlike to that which has just been mentioned. The equality of solids, it is natural to expect, must be proved like the equality of plane figures, by shewing that they may be made to coincide, or to occupy the same space. But, though it be true that all solids which can be shewn to coincide are equal and similar, yet it does not hold conversely, that all solids which are equal and similar can be made to coincide. Though this assertion will, perhaps, at first, appear a paradox, yet the proof of it is extremely simple.

Let ABC be an isosceles triangle, of which the equal sides are AB and AC; from A draw AE perpendicular to the base BC, and BC will be bisected in E. From E draw ED perpendicular to the plane ABC, and from D, any point in it, draw DA, DB, DC to the three angles of the triangle ABC. The pyramid DABC is divided into two pyramids DABE, DACE, which, though no one will dispute their equality, cannot be so applied to one another as to coincide. For, though the triangles ABE, ACE are equal, BE being equal to CE, EA common to both, and the angles AEB, AEC equal, because they are



right

Book VII. right angles, yet if these two triangles be applied to one another, so as to coincide, the solid DACE will, nevertheless, as is evident, fall without the solid DABE, for the two solids will be on the opposite sides of the plane ABE. In the same way, though all the planes of the pyramid DABE may easily be shewn to be equal to those of the pyramid DACE, each to each; yet will the pyramids themselves never coincide, though the equal planes be applied to one another, because they are on the opposite sides of those planes.

It may be said, then, on what ground do we conclude the pyramids to be equal? The answer is, because their construction is entirely the same, and the conditions that determine the magnitude of the one, identical with those that determine the magnitude of the other. For the magnitude of the pyramid DABE is determined by the magnitude of the triangle ABE, the length of the line ED, and the position of ED, in respect of the plane ABE; three circumstances that are precisely the same in the two pyramids, so that there is nothing that can determine one of them to be greater than another.

This reasoning appears perfectly conclusive and satisfactory, and it seems also very certain, that there is no other principle on which the relation of the solids DABE, DACE to one another, can be determined. Neither is this a case that occurs rarely; it is one, that in the comparison of magnitudes having three dimensions, presents itself continually; for, though two plane figures that are equal and similar can always be made to coincide, yet, with regard to solids that are equal and similar, if they have not a certain regularity in their construction, there will be found just as many cases in which they cannot, as in which they can coincide. Even figures described on surfaces, if they are not plane surfaces, may be equal and similar without the possibility of coinciding. Thus, in the figure described on the surface of a sphere, called a spherical triangle, if we suppose it to be isosceles, and a perpendicular to be drawn from the vertex on the base, it will not be doubted, that it is thus divided into two right angled spherical triangles equal and similar to one another, and which, nevertheless, cannot be so laid on one another as to agree. The same holds in innumerable other instances, and

and therefore it is evident, that the Axioms at present admitted into Geometry are not sufficient for the comparison of all geometrical magnitudes, and that there is a principle, more general, and fundamental than that of the equality of coinciding figures, which ought to be introduced. What this principle is has also appeared very clearly in the course of these remarks; and it is indeed no other than the principle so celebrated in the philosophy of Leibnitz, under the name of THE SUFFICIENT REASON. For it was shewn, that the pyramids DABE and DACE are concluded to be equal, because each of them is determined to be of a certain magnitude, rather than of any other, by conditions that are the same in both, so that there is no REASON for the one being greater than the other. This Axiom may be rendered general by saying, That things of which the magnitude is determined by conditions that are exactly the same, are equal to one another; or, it might be expressed thus: Two magnitudes A and B are equal, when there is no reason that A should exceed B, rather than that B should exceed A. Either of these will serve as the fundamental principle for comparing geometrical magnitudes of every kind; they will apply in those cases where the coincidence of magnitudes with one another has no place; and they will apply with great readiness, to the cases in which a coincidence may take place, such as in the 4th, the 8th, or the 26th of the First Book of the Elements.

The only objection to this Axiom is, that it is somewhat of a metaphysical kind, and belongs to the doctrine of the *sufficient reason*, which is looked on with a suspicious eye by some philosophers. But this is no solid objection; for such reasoning may be applied with the greatest safety to those objects with the nature of which we are perfectly acquainted and of which we have complete definitions, as in pure mathematics. In physical questions, the same principle cannot be applied with equal safety, because in such cases we have seldom a complete definition of the things we reason about, or one that includes in it all their properties. Thus, when Archimedes proved the spherical figure of the earth, by reasoning on a principle of this sort, he was led to a false conclusion, because he knew nothing of the rotation of the earth on its axis, which places the particles of that body, though at equal distances from the centre, in circumstances very different from one another. But, concerning those things that are the creatures

Book VII. creatures of the mind altogether, like the objects of mathematical investigation, there can be no danger of being misled by the principle of the sufficient reason, which at the same time furnishes us with the only single Axiom, by help of which we can compare together geometrical quantities, whether they be of one, of two, or of three dimensions.

From the remarks just made, it necessarily follows, that some demonstrations relating to the comparison of solids that have hitherto been reckoned unexceptionable, are not conclusive, at least, not so generally so, as the enunciations import. Such is the Prop. C, which Dr Simson has introduced into the 11th of Euclid, with a view of supplying the defects of Euclid's method. The proposition is, that "solid figures contained by the same number of equal and similar planes, alike situated, and having none of their solid angles contained by more than three plane angles, are equal and similar to one another." Dr Simson proves the truth of this proposition by placing the solids on the same base, and so making them to coincide. This, however, cannot always be done with solids that are similar and equal, as has been shewn, so that the proposition is necessarily limited in its extent, and that limitation ought not to be understood as implied in the words, alike situated. For this phrase is applied to plane figures, in which, when they are similar and equal, a coincidence may always be supposed; and it certainly signifies in that case nothing but that the sides are placed in the same order, and that the angles they make with one another, are directed the same way with respect to the figure in each; that is, if any angle is exterior, or *re-entering* in the one, it is the same in the other, and so on. It is in the same way that it should be understood when applied to solids.

For the same reason, the 21st proposition of the seventh Book of these Elements is not demonstrated in its full extent. This, however, does not affect the subsequent reasoning, because the case which is demonstrated is sufficient for the proof of the propositions that follow.

Another remark concerning the geometry of solids is, that Euclid, and the mathematicians who followed him, were in the wrong, when they held, that those solids are equal which
are

are contained under the same number of equal and similar planes, Book VII.
without adding, that the planes must be similarly situated. }
For this remark we are indebted to Dr Simson, who has shewn,
that unless the condition, of the planes being similarly situated,
be added, the proposition is false, in so much that innumerable
solids may exist, contained by the same number of similar
and equal planes, that shall be all unequal to one another.
He has proved the same with respect to solid angles; which
were before held to be equal when they were contained
by the same number of equal plane angles; for he shews
that there, too, unless the plane angles be similarly situated,
the solid angles may be unequal.

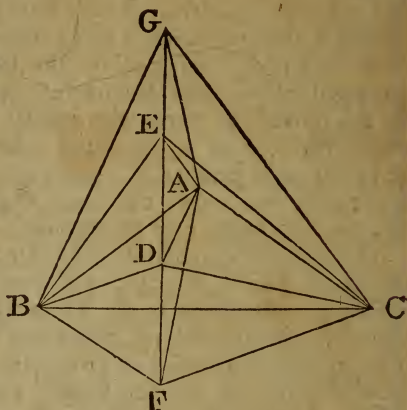
This remark was published by Dr Simson, in the first edition of his Euclid, in 1756, on the very same year when M. le Sage communicated to the Academy of Sciences the observation on the same subject mentioned in the last note; and it is singular, that these two geometers, without any communication with one another, should, almost at the same time, have made two discoveries so nearly connected with one another, and yet, that neither of them should have perceived the whole of the truth, so that the discovery of each is incomplete without that of the other.

When we speak of the planes of two solids being similarly situated, it is always meant, that the planes are in the same order, and their inclinations directed the same way in respect of the solid. Dr Simson has proved the truth of his remark as follows:

“ Let there be any plane rectilinear figure, as the triangle ABC, and from a point D within it, draw the straight line DE at right angles to the plane ABC; in DE take DE, DF equal to one another, upon the opposite sides of the plane, and let G be any point in EF; join DA, DB, DC; EA, EB, EC; FA, FB, FC; GA, GB, GC: Because the straight line EDF is at right angles to the plane ABC, it makes right angles with DA, DB, DC, which it meets in that plane; and in the triangles EDB, FDB, ED and DB are equal to FD and DB, each to each, and they contain right angles; therefore the base EB is equal to the base FB; in the same manner EA is equal to FA, and EC to FC: And in the triangles EBA, FBA, EB, BA are equal to FB, BA, and the base EA is equal to the base FA; wherefore the angle EBA is equal

Book VII. equal to the angle FBA, and the triangle EBA equal to the triangle FBA, and the other angles equal to the other angles; therefore these triangles are similar:

In the same manner the triangle EBC is similar to the triangle FBC, and the triangle EAC to FAC; therefore there are two solid figures, each of which is contained by six triangles, one of them by three triangles, the common vertex of which is the point G, and their bases the straight lines AB,



BC, CA, and by three other triangles the common vertex of which is the point E, and their bases the same lines AB, BC, CA: The other solid is contained by the same three triangles, the common vertex of which is G, and their bases AB, BC, CA; and by three other triangles, of which the common vertex is the point F, and their bases the same straight lines AB, BC, CA: Now the three triangles GAB, GBC, GCA are common to both solids, and the three others EAB, EBC, ECA of the first solid have been shown equal and similar to the three others FAB, FBC, FCA of the other solid, each to each; therefore, these two solids are contained by the same number of equal and similar planes: But that they are not equal is manifest, because the first of them is contained in the other: Therefore it is not universally true, that solids are equal which are contained by the same number of equal and similar planes."

"COR. From this it appears, that two unequal solid angles may be contained by the same number of equal plane angles."

"For the solid angle at B, which is contained by the four plane angles EBA, EBC, GBA, GBC is not equal to the solid angle at the same point B, which is contained by the four plane angles FBA, FBC, GBA, GBC; for this last contains

the other: And each of them is contained by four plane angles, which are equal to one another, each to each, or are the self same, as has been proved: And indeed, there may be innumerable solid angles all unequal to one another, which are each of them contained by plane angles that are equal to one another, each to each: It is likewise manifest, that the before mentioned solids are not similar, since their solid angles are not all equal."

P R O P. XXXIII.

The principal departure from the method of Euclid, introduced into this Book, respects the pyramid; in treating of which, the method of exhaustions is first employed, as in this proposition. The demonstration used here has the advantage of applying very generally to all the solids which require that method, and which can be treated of in the Elements; and it has also a very close and evident affinity to the methods used in the higher geometry.

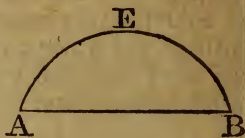
BOOK VIII.

THE object of this Book being to compare with one another, and with the parallelepiped, the solids that depend on the circle, it is necessary to begin with the quadrature of that figure, or the comparison of the space contained within it with rectilinear figures. The first eight propositions relate entirely to this subject, and contain the rules both for finding the length of the circumference nearly, and also the space contained within it, and in demonstrating them, I have followed partly the method of Thomas Simpson, in his Elements, and partly that of Archimedes, in his *Dimensio circuli*. As the foundation of the proposition for finding the

D d length

Book VIII.

length of the circumference, an Axiom is prefixed to the beginning of this Book, the same in effect with that which Archimedes employs as the basis of his investigation, though somewhat more concisely expressed. I have already taken notice of this Axiom, in the notes on parallel lines, (p. 372.) as a proposition which it has been found impossible to demonstrate, though it certainly might be expected to follow necessarily from the ideas of a straight, and a curve line. Another thing remarkable concerning this Axiom is, that in the simplest case of it, when the figures are rectilineal, it admits of demonstration; for it then coincides with the 20th and 21st of the first of Euclid, in the former of which it is shewn, that any two sides of a triangle are greater than the third, and in the latter, that of two triangles which have the same base, that which is within the other, has the sum of its sides the least. But when, instead of two sides of a triangle, we have a curve AEB, and a straight line AB terminated in the same points A and B, it then ceases to be possible to prove, that AEB is greater than AB; as also, that any curve line which includes AEB, and is terminated in A and B, is greater than AEB; and the truth of both these propositions must be taken for granted. At least, after Archimedes has considered them as Axioms, we may reasonably despair of ever seeing them demonstrated. Here, therefore, we find ourselves under the necessity of taking for granted a general proposition, after having demonstrated the simplest case of it. For all this, the trouble taken, to demonstrate that case, ought not to be considered as superfluous, for we are thereby prepared to admit the general Axiom, and have the satisfaction of observing its perfect agreement with truths, previously demonstrated. It is probably from the circumstance of a curve line having no definition, but one which is merely negative, viz. that it is a line of which no part is a straight line, that the impossibility of demonstrating this proposition takes its rise.



P R O P.

PROP. IV. V. &c.

The demonstrations of the 5th and 6th propositions require the method of exhaustions, that is to say, they prove a certain property to belong to the circle, because it belongs to the rectilinear figures inscribed in it, or described about it according to a certain law, in the case when those figures approach to the circle so nearly as not to fall short of it, or to exceed it by any assignable difference. This principle is general, and is the only one by which we can possibly compare curvelineal, with rectilinear spaces, or the length of curve lines with the length of straight lines, whether we follow the methods of the ancient or of the modern geometers. It is, therefore, a great injustice to the latter methods to represent them as standing on a foundation less secure than the former; they stand in reality on the same, and the only difference is, that the application of the principle, common to them both, is more general and expeditious in the one case than in the other. This identity of principle, and affinity of the methods used in the elementary and the higher mathematics, it seems the more necessary to observe, that some learned mathematicians have appeared not to be sufficiently aware of it, and have even endeavoured to demonstrate the contrary. An instance of this is to be met with in the preface of the valuable edition of the works of Archimedes, lately printed at Oxford. In that preface, Torelli, the learned commentator, whose labours have done so much to elucidate the writings of the Greek geometer, but who is so unwilling to acknowledge the merit of the modern analysis, undertakes to prove, that it is impossible from the relation which the rectilinear figures inscribed in, and circumscribed about, a given curve, have to one another, to conclude any thing concerning the properties of the curvelineal space itself, except in certain circumstances which he has not precisely described. With this view he attempts to shew, that if we are to reason from

Book VIII. the relation which certain rectilineal figures belonging to the circle have to one another, notwithstanding that those figures may approach so near to the circular spaces within which they are inscribed, as not to differ from them by any assignable magnitude, we shall be led into error, and shall seem to prove, that the circle is to the square of its diameter exactly as 3 to 4. Now, as this is a conclusion which the discoveries of Archimedes himself, prove so clearly to be false, Torelli argues, that the principle from which it is deduced must be false also; and in this he would no doubt be right, if his former conclusion had been fairly drawn. But the truth is, that a very gross paralogism is to be found in that part of his reasoning, where he makes a transition from the ratios of the small rectangles, inscribed in the circular spaces, to the ratios of the sums of those rectangles, or of the whole rectilineal figures. In doing this, he takes for granted a proposition which, it is wonderful, that one who had studied geometry in the school of Archimedes, should for a moment have supposed to be true. The proposition taken in the simplest view of it is this: If A, B, C, D, E, F, be any number of magnitudes, and a, b, c, d, e, f, as many others; and if $A : B :: a : b$,

$$C : D :: c : d,$$

$E : F :: e : f$, then the sum of A, C and E will be to the sum of B, D and F, as the sum of a, c and e, to the sum of b, d and f; or $A+C+E : B+D+F :: a+c+e : b+d+f$. Now, this proposition, which Torelli supposes to be perfectly general, is not true, except in two cases, viz. either first, when

$$A : C :: a : c, \text{ and}$$

$$A : E :: a : e; \text{ and consequently,}$$

$$B : D :: b : d, \text{ and}$$

$B : F :: b : f$; or, secondly, when all the ratios of A to B, C to D, E to F, &c. are equal to one another. To demonstrate this, let us suppose that there are four magnitudes, and four others,

$$\text{thus, } A : B :: a : b, \text{ and}$$

$C : D :: c : d$, then we cannot have $A+C : B+D :: a+c : b+d$, unless either, $A : C :: a : c$, and $B : D :: b : d$; or $A : C :: b : d$, and consequently $a : b :: c : d$.

Take a magnitude K, such that $a : c :: A : K$, and another L, such that $b : d :: B : L$; and suppose it true, that $A+C : B+D :: a+c : b+d$. Then, because by inversion, $K : A :: c : a$, and, by hypothesis, $A : B :: a : b$, and also $B : L :: b : d$, *ex æquo*, $K : L :: c : d$; and consequently, $K : L :: C : D$.

K, A, B, L.
c, a, b, d.

Again, because $A : K :: a : c$, by addition,
 $A+K : K :: a+c : c$; and, for the same reason,
 $B+L : L :: b+d : d$, or, by inversion,
 $L : B+L :: d : b+d$. And, since it has been
shewn, that $K : L :: c : d$; therefore, *ex æquo*,

A+K, K, L, B+L.
a+c, c, d, b+d.

$A+K : B+L :: a+c : b+d$; but by hypothesis,
 $A+C : B+D :: a+c : b+d$, therefore
 $A+K : A+C :: B+L : B+D$.

Now, first, let K and C be supposed equal, then, it is evident, that L and D are also equal; and therefore, since by construction $a : c :: A : K$, we have also $a : c :: A : C$; and, for the same reason, $b : d :: B : D$, and these analogies form the first of the two conditions, of which one is affirmed above to be always essential to the truth of Torelli's proposition.

Next, if K be greater than C, then, since
 $A+K : A+C :: B+L : B+D$, by division,
 $A+K : K-C :: B+L : L-D$. But, as was shewn,
 $K : L :: C : D$, by conversion and alternation,
 $K-C : K :: L-D : L$, therefore, *ex æquo*,
 $A+K : K :: B+L : L$, and, lastly, by division,
 $A : K :: B : L$, or $A : B :: K : L$, that is,
 $A : B :: C : D$.

Wherefore, in this case the ratio of A to B is equal to that of C to D, and consequently, the ratio of a to b equal to that of

Book VIII. of c to d . The same may be shewn, if K is less than C ; therefore, in every case there are conditions necessary to the truth of Torelli's proposition, which he does not take into account, and which, as is easily shewn, do not belong to the magnitudes to which he applies it.

In consequence of this, the conclusion that is meant to be established respecting the circle falls entirely to the ground, and with it the more general inference that was aimed against the modern analysis.

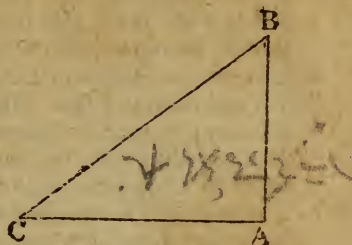
It will not, I hope, be imagined, that I have taken notice of these circumstances with any design to lessen the reputation of the learned Italian, who has in so many respects deserved well of the mathematical sciences, or to detract from the value of a posthumous work, which, by its elegance and correctness, does so much honour to the English editors. But I would warn the student against that spirit of party, which seeks to introduce itself even into the investigations of geometry, and to persuade us, that elegance, and even truth, are qualities possessed exclusively by the ancient methods of demonstration. The high tone in which Torelli censures the modern mathematics, is the more calculated to produce an opinion of this kind, that it is assumed by one who had studied the writings of Archimedes with so much diligence; and therefore, the observations, that have been made above, may be useful, by teaching us to listen with caution to his decisions.

P R O P. VII.

This proposition is the foundation of the application of arithmetic to geometry, and may be said to be the truth that connects together these two branches of the mathematics, though, in elementary treatises, I think, it has for the most part been omitted. In no case do we compute, in numbers, the area of any figure, or even calculate the length of a side of a triangle, from the other parts which determine it being given, without having recourse to this Theorem. If, for instance, from having two sides of a right angled triangle as ABC expressed in numbers, we would compute the length
of

of the remaining side BC, we must make use of this propo- Book VIII.
 sition, as well as of the 47th of

the first Book. If, for instance, $AB=6$, and $AC=8$, and, if it be required to find BC; then, since AB is to AC as 6 to 8, the square of AB will be to the square of AC, by this proposition, as 6×6 to 8×8 , or as 36 to 64, and therefore, also the sum of the squares on AB and AC will be to the square of AB as 100 to 36.



But the squares of AB and AC are equal to the square of BC; therefore, the square of BC is to the square of AB as 100 to 36, and therefore, by the second corollary, BC is to BA as 10 to 6. It is the same in other cases; and, though the steps may not all be taken so regularly as is done here, they are nevertheless as certainly implied.

P R O P. VIII.

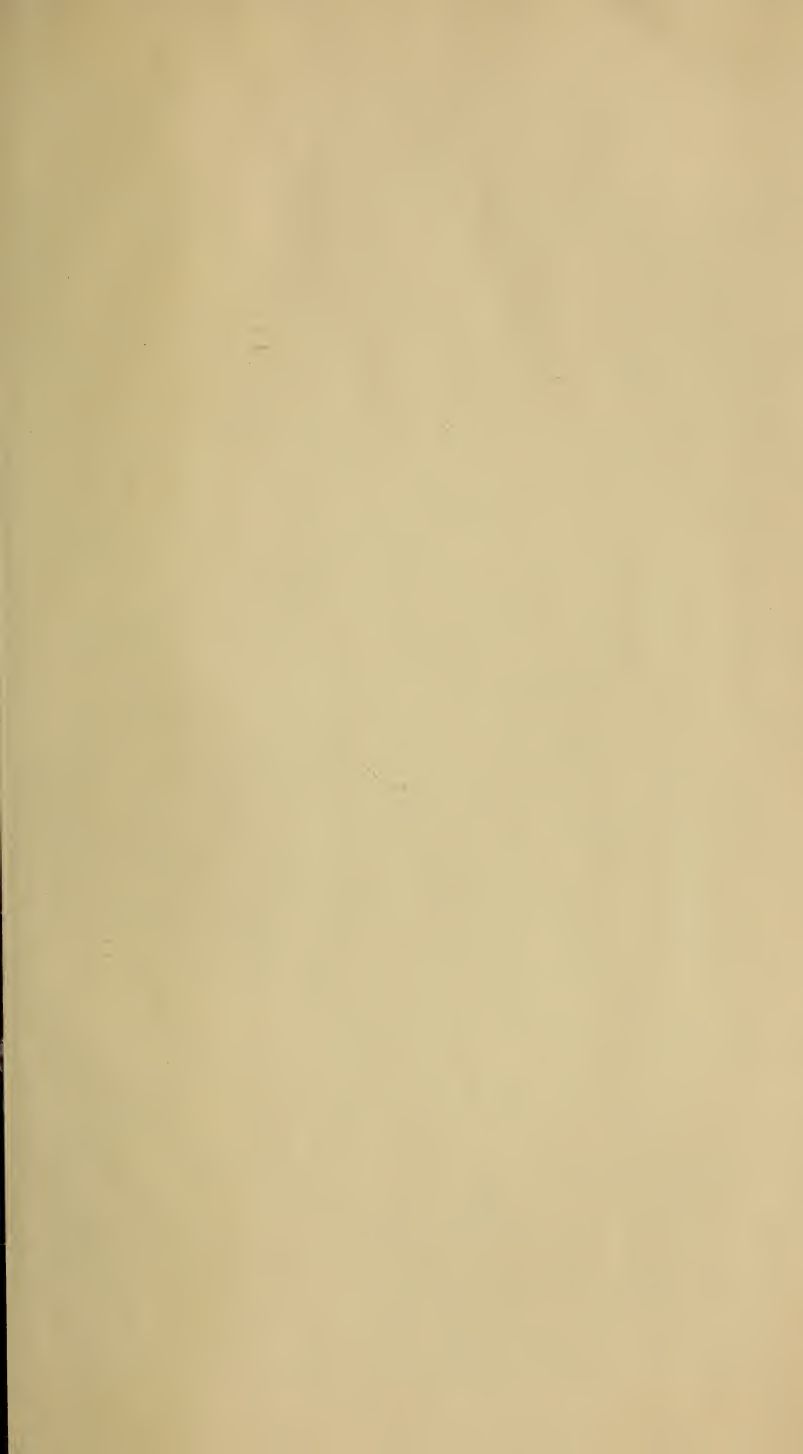
This enunciation is the same with that of the third of the *Dimensio Circuli* of Archimedes; but the demonstration is different, though it proceeds, like that of the Greek geometer, by the continual bisection of the 6th part of the circumference. In this proposition a particular notation is used, by putting the sign + after a number, to denote that something is to be added to it, and the sign —, to denote the contrary. Thus, when it is said, that AQ is to AD as $866.0254+$ to 1000, this signifies, that AQ is to AD as a number greater than 866.0254 is to 1000, and it is the same thing with saying, that AQ has to AD a greater ratio than 866.0254 to 1000. In the same manner, when it is demonstrated in the second part of the proposition, that LN is to CD as $32.71927-$ to 1000, it is meant, that LN is to CD as a number less than 32.71927 to 1000, or that LN has to CD a less ratio than that of 32.71927 to 1000.

The advantage that results from this mode of expression is, that when we are to reason about the sums or differences of two lines, of which the ratio is thus expressed, it can be done by

Book VIII. by the simple addition or division of proportionals, without any new proposition. Thus, when it is shewn that AQ , or which is the same, DH is to DA as $866.0254+$ to 1000 , it is evident from the 18th of the 5th, that DH and DA together, that is, AH is to DA as $1866.0254+$ to 1000 ; whereas, if it had been said that DH is to DA in a greater ratio than 866.0254 to 1000 , a separate proposition, or *lemma*, must have been introduced, for the express purpose of proving that HA has to DA a greater ratio than 1866.0254 to 1000 .

It must also be attended to, in reading this proposition, that the second figure is a part of the first, represented on a larger scale, for the sake of rendering the small divisions of the circumference, and the lines belonging to them, more distinct.

F I N I S.







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